A SEMIREGULARITY MAP FOR MODULES AND APPLICATIONS TO DEFORMATIONS

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Abstract. We construct a general semiregularity map for algebraic cycles as asked for by S. Bloch [Blo] in 1972. The existence of such a semiregularity map has well known consequences for the structure of the Hilbert scheme and for the variational Hodge conjecture. Aside from generalizing and extending considerably previously known results in this direction, we give new applications to deformations of modules that encompass, for example, results of Artamkin [Art] and Mukai [Muk].

The formation of the semiregularity map here involves powers of the cotangent complex, Atiyah classes, and trace maps, and is defined not only for subspaces of manifolds but for perfect complexes on arbitrary complex spaces. It generalizes in particular Illusie's [Ill] treatment of the Chern character to the analytic context and specializes to Bloch's earlier description of the semiregularity map for locally complete intersections as well as to the infinitesimal Abel-Jacobi map for submanifolds.

Contents

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1. Introduction

Let Z be a closed complex subspace of a compact complex manifold X and let $[Z]$ denote the corresponding point in the Douady space H_X of X, the complex analytic analogue of the Hilbert scheme. It is a classical fact that the tangent space of H_X at $[Z]$ is naturally isomorphic to $H^0(Z, \mathcal{N}_{Z/X})$, where $\mathcal{N}_{Z/X} = \mathcal{H}om_X(\mathcal{J}, \mathcal{O}_Z)$ denotes the normal sheaf of Z in X with $\mathcal{J} \subseteq \mathcal{O}_X$ the ideal sheaf of Z. Moreover, if Z is locally a complete intersection in X then the vanishing of $H^1(Z, \mathcal{N}_{Z/X})$ implies that $[Z]$ is a smooth point of H_X. It was, however, already observed by Severi [Sev] that the converse is not true in general. He introduced the notion of a semiregular curve on a surface to mean that the restriction map $H^0(X, \omega_X) \to H^0(Z, \omega_X|Z)$ is surjective or, dually, that the *semiregularity map* $H^1(Z, \mathcal{N}_{Z/X}) \to H^2(X, \mathcal{O}_X)$ is injective, and showed that the point of the Hilbert scheme corresponding to a semiregular curve is always smooth. This result was extended to divisors in arbitrary projective complex manifolds by Kodaira and Spencer [KSp]. In 1972, S. Bloch [Blo] was able to define more generally for every locally complete intersection Z of codimension q in X a semiregularity map $\tau : H^1(Z, \mathcal{N}_{Z/X}) \to H^{q+1}(X, \Omega_X^{q-1})$ to show again that the injectivity of τ implies that $[Z]$ is a smooth point of H_X. His semiregularity map admits a simple description using Serre duality. However, in case of an arbitrary subspace Z of X the obstructions for extending embedded deformations lie in the tangent cohomology group $T^2_{Z/X}(\mathcal{O}_Z)$, and it is no longer possible to apply duality. Thus the problem arises to define such a semiregularity map by other means.

In our approach we will more generally assign first a semiregularity map $\sigma : \operatorname{Ext}^2_X(\mathcal{F}, \mathcal{F}) \to \prod_{q\geq 0} H^{q+2}(X, \Omega^q_X)$ to every coherent $\mathcal{O}_X$–module $\mathcal{F}$ on a complex manifold. Indeed this map will be defined for any coherent $\mathcal{O}_X$–module $\mathcal{F}$ of locally finite projective dimension, or even for perfect complexes of modules, on arbitrary complex spaces, and it occurs as the component $\sigma = \sigma^{(2)}$ of a family of maps

$$\sigma^{(r)} : \operatorname{Ext}^r_X(\mathcal{F}, \mathcal{F}) \to \prod_{q\geq 0} H^{q+r}(X, \Lambda^q \mathbb{L}_X), \quad r \geq 0,$$

where $\mathbb{L}_X$ denotes the cotangent complex of X. We will outline in brief our construction when X is smooth, which special case is also subject of our survey [BFl2].

To begin with, we assign to $\mathcal{F}$ its Atiyah class, as originally defined in [At] for locally free sheaves. Following [Ill], a possible way of construction for any coherent $\mathcal{O}_X$–module is to use the extension on $X \times X$ that defines the module of analytic differential forms,

$$0 \to \mathcal{J}/\mathcal{J}^2 \cong \Omega^1_X \to \mathcal{O}_{X\times X}/\mathcal{J}^2 \to \mathcal{O}_X \to 0, \tag{1}$$

where $\mathcal{J} \subseteq \mathcal{O}_{X\times X}$ is the ideal of the diagonal. With $p_i : X \times X \to X$ the i^{th} projection for $i = 1, 2$, we tensor (1) with $p_1^*(\mathcal{F})$ and consider the resulting sequence

$$0 \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_X \to p_1^*(\mathcal{F}) \otimes_{\mathcal{O}_{X\times X}} \mathcal{O}_{X\times X}/\mathcal{J}^2 \to \mathcal{F} \to 0$$

as an extension of $\mathcal{O}_X$–modules via p_{2*} so that it defines an element

$$\operatorname{At}(\mathcal{F}) \in \operatorname{Ext}^1_X(\mathcal{F}, \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_X),$$

the *Atiyah class* of $\mathcal{F}$. Taking powers gives elements $\operatorname{At}^q(\mathcal{F}) \in \operatorname{Ext}^q_X(\mathcal{F}, \mathcal{F} \otimes \Omega^q_X)$. Now the q^{th} component of the semiregularity map σ is the composition of the two maps

$$\sigma_q : \operatorname{Ext}^2_X(\mathcal{F}, \mathcal{F}) \xrightarrow{*\cdot(-\operatorname{At}(\mathcal{F}))^q/q!} \operatorname{Ext}^{q+2}_X(\mathcal{F}, \mathcal{F} \otimes \Omega^q_X) \xrightarrow{\operatorname{Tr}} H^{q+2}(X, \Omega^q_X),$$

where Tr is the trace map as defined in [Ill, OTT]. Finally, to get a semiregularity map for subspaces $Z \subseteq X$, we observe that there is a natural homomorphism from $T^k_{Z/X}(\mathcal{O}_Z)$ into $\mathrm{Ext}^k_X(\mathcal{O}_Z, \mathcal{O}_Z)$ for each $k \geq 0$. Composing this for $k = 2$ with the map σ above in case $\mathcal{F} = \mathcal{O}_Z$ gives the desired semiregularity map

$$\tau = (\tau_q)_{q\geq 0} : T^2_{Z/X}(\mathcal{O}_Z) \to \prod_{q\geq 0} H^{q+2}(X, \Omega^q_X)$$

for subspaces. We will verify in Section 8 that for a locally complete intersection Z in X of codimension q the component τ_{q-1} of our semiregularity map coincides with the classical one defined by Bloch.

To understand some of the geometrical implications of such semiregularity maps, let us restrict further to the case of coherent modules on a compact algebraic manifold, the case of subspaces being similar. With respect to the Hodge decomposition, $H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^q(X, \Omega^p_X)$, the Chern character of a coherent sheaf is obtained from its Atiyah class by the formula

$$\mathrm{ch}(\mathcal{F}) = \mathrm{Tr}\exp(-\mathrm{At}(\mathcal{F})) = \sum_{k\geq 0} \frac{(-1)^k}{k!} \mathrm{Tr}(\mathrm{At}^k(\mathcal{F}))\,,$$

see [At] for the case of vector bundles and [Ill, OTT] for the general case.

Assume now given an infinitesimal deformation of X represented by a class $\xi \in H^1(X, \Theta_X)$, where Θ_X denotes the tangent bundle. By Bloch's interpretation of Griffiths' transversality theorem, for fixed $k \geq 0$ the unique horizontal extension of the cohomology class $\mathrm{ch}_{k+1}(\mathcal{F})$ relative to the Gauß-Manin connection stays of Hodge type $(k+1, k+1)$ if and only if the contraction of this class by ξ vanishes, $\langle \xi, \mathrm{ch}_{k+1}(\mathcal{F})\rangle = 0$ in $H^{k+2}(X, \Omega^k_X)$. On the other hand, let us consider the deformations of $\mathcal{F}$ itself instead of just extending its Chern character horizontally. The deformations of $\mathcal{F}$ are controlled by the differential graded Lie algebra underlying $\mathrm{Ext}^\bullet_X(\mathcal{F}, \mathcal{F})$ so that the space of infinitesimal deformations is given by $\mathrm{Ext}^1_X(\mathcal{F}, \mathcal{F})$ and the obstructions to extend deformations live in $\mathrm{Ext}^2_X(\mathcal{F}, \mathcal{F})$. Contracting against the negative of the Atiyah class serves as an obstruction map

$$\mathrm{ob} := \langle *, -\mathrm{At}(\mathcal{F})\rangle : H^1(X, \Theta_X) \longrightarrow \mathrm{Ext}^2_X(\mathcal{F}, \mathcal{F})$$

so that $\mathcal{F}$ admits a deformation into the direction of ξ if and only if $\mathrm{ob}(\xi) = 0$; see [Ill] for the algebraic and 4.4 for the analytic case. A key observation is now that the maps just described fit into a commutative diagram

$$\begin{tikzcd} H^1(X, \Theta_X) \arrow[rr, "{\langle *, -\mathrm{At}(\mathcal{F})\rangle}"] \arrow[dr, "{\langle *, \mathrm{ch}_{k+1}(\mathcal{F})\rangle}"'] & & \mathrm{Ext}^2_X(\mathcal{F}, \mathcal{F}) \arrow[dl, "\sigma_k"] \\ & H^{k+2}(X, \Omega^k_X)\,. & \end{tikzcd}$$

As a consequence of (a suitable generalization) of this fact and and generalizing the arguments of [Blo] we obtain for instance in Section 5 that *the variational Hodge conjecture holds for cycles that are representable as* $(k+1)^{st}$ *component of the Chern character of a* k*-semiregular sheaf* $\mathcal{F}$, where we mean by k*-semiregular* that the component σ_k of the semiregularity map for $\mathcal{F}$ is injective. An analogous result holds for subspaces $Z \subseteq X$ that are (k-)semiregular in the corresponding sense.

Other important applications are to deformations of modules. In analogy with the aforementioned results of Bloch we will show that *the basis of the semiuniversal*

deformation of $\mathcal{F}$ is smooth if the semiregularity map σ is injective. We will deduce this result more generally for arbitrary singular complex spaces X and also for relative situations. We derive as well analogous smoothness criteria for the Douady space and the Quot-space, and give applications to deformations of holomorphic mappings.

Ideally, the semiregularity map, say, for a module $\mathcal{F}$ should correspond to a morphism between two deformation theories so that it maps the obstruction space $\mathrm{Ext}^2_X(\mathcal{F}, \mathcal{F})$ into the obstruction space of some other deformation theory. It seems quite clear that this second deformation theory should be given in terms of the intermediate Jacobians, or, more naturally, by Deligne cohomology. The map from the deformations of $\mathcal{F}$ into the intermediate Jacobian, say $J^k(X)$, should be a generalized Abel-Jacobi map that associates to a deformation of $\mathcal{F}$ over a germ $(S, 0)$ the map of S into the intermediate Jacobian given by integration over a topological cycle whose boundary is the difference of the k^{th} Chern characters of the special fibre and the fibre over s. As the intermediate Jacobian is smooth this would provide a satisfactory explanation of the fact that the injectivity of σ implies the smoothness of the versal deformation of $\mathcal{F}$, and it would show that all obstructions of $\mathcal{F}$ vanish under σ and not merely the curvilinear ones as we show here; for the special case of submanifolds instead of modules see [Cle, Ran4]. Such an interpretation indeed applies for the lowest component of the semiregularity map: the work of Artamkin [Art] and Mukai [Muk] interprets $\sigma_0 : \mathrm{Ext}^2_X(\mathcal{F}, \mathcal{F}) \to H^2(X, \mathcal{O}_X)$ as the map between obstruction spaces for the deformations of $\mathcal{F}$ versus those of its determinant line bundle. As a further clue that such interpretation might be true in general we verify in Section 9 that for a submanifold Z of X the differential of the Abel-Jacobi map admits the same homological description as the semiregularity map.

A few remarks about the contents of the various sections: In Section 2 we review the technique of Forster-Knorr systems, originally used in [FKn] and further exploited in [Pal, Fle1], to construct a cotangent complex for complex spaces. Most of that material is a largely generalized version of parts of [Fle1]. As this source is not easily accessible we use the opportunity to give an exposition of that technique of simplicial spaces of Stein compact sets in the generality we need. Following [Pal] we will review in brief the notion of resolvents and give the relevant descriptions of tangent (co)homology as used later on.

In Section 3 we construct the Atiyah class of a coherent sheaf $\mathcal{F}$ as a class in $\mathrm{Ext}^1_X(\mathcal{F}, \mathcal{F} \underline{\otimes} \mathbb{L}_X)$, thereby generalizing the construction by Illusie [Ill] to the analytic case. First we do this for modules on simplicial spaces of Stein compact sets and then use the results of the previous section to descend to actual complex spaces. Following the classical approach of Atiyah [At], see also [ALJ], we will construct these classes using connections, in our case on modules over the resolvent of a complex space, thus verifying the basic funtorial properties by explicit computation.

Section 4 contains the construction of the semiregularity map for modules as well as for subspaces. We give an interpretation of the obstruction map for modules or subspaces in terms of Atiyah classes and derive the aforementioned commutative diagram. This is the basic tool in Section 5, where we prove the variational Hodge conjecture for the special case described above.

In Section 6 we prove some general criteria for the smoothness of the basis of a semiuniversal deformation. We give a new and transparent proof of the T^1-lifting criterion of Ran and Kawamata [Kaw1, Kaw2, Ran2, Ran3] and show how

to deduce their results by a simple argument from the well known Jacobian criterion for smoothness. Our method also yields new generalizations to the relative case. These results, together with the existence of the semiregularity map, have many applications in deformation theory, some of which we treat in Section 7. In the first part we deduce applications to deformations of modules as mentioned above. In the second part we turn to the Douady space and give various criteria for its smoothness. The more general case of the Quot-scheme is treated in the third part, and in the last part we apply our constructions to deformations of mappings and define a semiregularity map under very general circumstances. For the special case of stable curves, results in this direction were independently obtained by K. Behrend and B. Fantechi.

In the final Section 8 we compare our semiregularity map with the one constructed by S. Bloch. This requires an explicit description of the trace map via a Cousin type resolution. Moreover, we show how the infinitesimal Abel-Jacobi map fits into this framework.

In an Appendix we collect some results on integral dependence and infinitesimal deformations of complex spaces that are needed in Section 6. Especially the (elementary) characterizations of the subspaces of T^1_X given by the curvilinear extensions, respectively by $\mathrm{Ext}^1_X(\Omega^1_X, \mathcal{O}_X)$, seem to be new.

General notation. We explain some notation used throughout this paper.

Categories are written in boldface and categories like **Sets** should need no further explanation. Whenever we talk about isomorphism classes of objects from a category **C**, it will be assumed that those classes form a set. Such a category **C** is sometimes called *essentially small*.

A germ of a (formal) complex space is denoted $(S, 0)$ or often simply S. As a rule, every germ has 0 as its basepoint, and the same symbol represents the (reduced) point. For a (formal) complex space X the sheaf of holomorphic functions is as usual denoted $\mathcal{O}_X$, whereas for a germ $S = (S, 0)$ the symbol $\mathcal{O}_S$ indicates the local ring $\mathcal{O}_{S,0}$ and $\mathfrak{m}_S$ its maximal ideal.

If X is a complex space then $\mathbf{Coh}(X)$ will be its category of coherent modules. Similarly, if $S = (S, 0)$ is the germ of a (formal) complex space, $\mathbf{Coh}(S)$, $\mathbf{Coh}_{art}(S)$ will denote the categories of finite, respectively finite artinian $\mathcal{O}_{S,0}$–modules. A closed embedding $S \hookrightarrow S'$ of complex spaces is an *extension* of S by $\mathcal{M} \in \mathbf{Coh}(S)$ if the ideal $\mathcal{I} := \mathrm{Ker}(\mathcal{O}_{S'} \to \mathcal{O}_S)$ defining S in S' is of square zero and isomorphic to $\mathcal{M}$ as $\mathcal{O}_S$–module under a fixed isomorphism. In particular, $S[\mathcal{M}]$ indicates the *trivial extension* whose structure sheaf $\mathcal{O}_{S[\mathcal{M}]}$ is the direct sum $\mathcal{O}_S \oplus \mathcal{M}\varepsilon$ endowed with the multiplication $(s + m\varepsilon)(s' + m'\varepsilon) = ss' + (sm' + ms')\varepsilon$ so that $\varepsilon^2 = 0$.

To reduce complexity of display, we use unadorned tensor products, such as $\mathcal{M} \otimes \mathcal{N}$, whenever the sheaf or ring over which the tensor product is taken should be clear from the context. We also use $\underline{\otimes}$ instead of $\otimes^{\mathbb{L}}$ to denote derived tensor products.

2. Homological algebra on simplicial schemes of Stein compact sets

Let X be a complex space and $\mathcal{F}$ a coherent $\mathcal{O}_X$–module. In the introduction we reviewed for X smooth the construction of the *Atiyah class*, $\mathrm{At}(\mathcal{F}) \in \mathrm{Ext}^1_X(\mathcal{F}, \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_X)$, of $\mathcal{F}$ that is given by the extension

$$0 \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_X \to p_1^*(\mathcal{F}) \otimes_{\mathcal{O}_{X\times X}} \mathcal{O}_{X\times X}/\mathcal{J}^2 \to \mathcal{F} \to 0\,.$$

In the singular case, however, we need to produce more generally the Atiyah class in

$$\operatorname{Ext}^1_X(\mathcal{F}, \mathcal{F} \underline{\otimes}_{\mathcal{O}_X} \mathbb{L}_X)\,,$$

where $\mathbb{L}_X$ is the cotangent complex of X and $\underline{\otimes}$ denotes the derived tensor product.

In the algebraic setting this was done by Illusie [Ill] using simplicial methods. Those do not immediately generalize to the analytic situation due to the lack of global resolutions. Instead we use the technique of Forster-Knorr systems [FKn] on simplicial schemes of Stein compact sets and construct resolvents on complex spaces as in [Fle1, Pal]. Alternatively one might be tempted to use the method of twisted cochains as developed in [OTT], but this theory is so far only available for manifolds, in particular the theory of cotangent complexes on singular spaces has not yet been established in that framework.

We first state and (indicate how to) prove the results on the homological algebra of Forster-Knorr systems that we will use. Key references are [Fle1], [Pal]; see also [BMi]. All our complex spaces are assumed to be paracompact.

2.1. A subset K of a complex space X is called *Stein compact* if it is compact, semianalytic and admits arbitrary small open neighbourhoods that are Stein. We equip K with the structure sheaf $\mathcal{O}_K := \mathcal{O}_X|_K$ so that the ring $\Gamma(K, \mathcal{O}_K)$ consists of all K-germs of functions that are analytic in an open neighbourhood of K in X. In the extreme case that $K = \{x\}$ consists just of a point in X one retrieves the local analytic algebra $\mathcal{O}_{X,x}$. Every coherent $\mathcal{O}_K$–module for a Stein compact set K satisfies Cartan's Theorems A and B, and by a fundamental result of Frisch [Fri] the ring $\Gamma(K, \mathcal{O}_K)$ is noetherian. These facts imply that a Stein compact set behaves like a noetherian affine scheme: for example, the category $\mathbf{Coh}\,\mathcal{O}_K$ of coherent $\mathcal{O}_K$–modules is equivalent to the category of finite $\Gamma(K, \mathcal{O}_K)$–modules, in particular it contains enough projectives. Again as for affine schemes, these projectives are usually not projective objects in the category of all $\mathcal{O}_K$–modules.

Note that the *dimension* of a Stein compact set is given by

$$\dim K := \sup\{\dim \mathcal{O}_{K,x} \mid x \in K\} = \inf\{\dim U \mid K \subseteq U \subseteq X, U \text{ open}\}\,.$$

If $K \subseteq X$, $L \subseteq Y$ are Stein compact sets with X, Y complex spaces over some complex space Z, then the product $K \times L \subseteq X \times Y$ as well as the fibre product $K \times_Z L \subseteq X \times_Z Y$ are again Stein compact sets.

A covering of a complex space X by Stein compact sets $\{X_i\}_{i\in I}$ is called *locally finite* if every point in X admits an open neighbourhood that meets only finitely many X_i. Any two such coverings clearly admit common refinements and, as X is paracompact, each covering by open sets can be refined to a locally finite covering by Stein compact sets.

2.2. Let X be a complex space over Y and $\{X_i\}_{i\in I}$ a locally finite covering of X by Stein compact sets. The *nerf* of the covering is the simplicial set

$$A = \{\alpha \subseteq I \mid X_\alpha := \bigcap_{i\in\alpha} X_i \neq \emptyset\}\,,$$

whose simplices are thus finite subsets of I, and

$$X_* = (X_\alpha)_{\alpha\in A}\,, \text{with the inclusions} \quad X_\beta \hookrightarrow X_\alpha \quad \text{for } \alpha \subseteq \beta,$$

forms the corresponding *simplicial scheme of Stein compact sets* over A. Denote $|\alpha| = k$ the dimension of the simplex $\alpha = \{i_0, ..., i_k\}$, not to be confused with $\dim X_\alpha$, the complex dimension of the Stein compact X_α.

Aside from simplicial schemes of Stein compact sets arising from locally finite coverings we will also need the following type. Let A be again a simplicial set of finite subsets of some index set I and assume given for any vertex $i \in I$ a Stein compact set $L_i \subseteq \mathbb{C}^{n_i} \times Y$ for some n_i. Set

$$W_\alpha := \prod\nolimits_{i\in\alpha}^{Y} L_i\,, \quad \alpha \in A\,,$$

where $\prod^Y$ denotes the fibre product over Y, and let the natural projections $p_{\alpha\beta}: W_\beta \to W_\alpha$ for $\alpha \subseteq \beta$ serve as transition maps. The natural maps from W_α onto Y are smooth and compatible with the transition maps. The resulting simplicial scheme $W_* \to Y$ over Y is called *smooth*.

Returning to the complex space X with its given covering choose closed Y-embeddings $X_i \hookrightarrow L_i \subseteq \mathbb{C}^{n_i} \times Y$ with L_i a Stein compact subset. These data yield diagonal embeddings $X_\alpha \hookrightarrow W_\alpha, \alpha \in A$, that define a morphism of simplicial schemes $X_* \hookrightarrow W_*$ over Y. We will refer to it as a (simplicial) *smoothing* of the given map $X \to Y$.

2.3. Now we consider (negatively graded, simplicial) DG algebras $\mathcal{R}_* = \bigoplus_{i\le 0} \mathcal{R}_*^i$ over W_*. The differential of such an algebra is a derivation of degree $+1$ and will be denoted $\partial_{\mathcal{R}_*}$ or simply ∂. The simplicial structure consists of a system of compatible maps $p_{\alpha\beta}^{-1}(\mathcal{R}_\alpha) \to \mathcal{R}_\beta$ for $\alpha \subseteq \beta$ that are morphisms of DG algebras over W_β. If f is a local homogeneous section of $\mathcal{R}_*$, then $|f|$ denotes its degree. All our DG algebras will be (graded) *commutative* so that the product on $\mathcal{R}_*$ satisfies the sign rule

$$fg = (-1)^{|f||g|} gf\,.$$

Graded modules over $\mathcal{R}_*$ are defined in the obvious way: An *$\mathcal{R}_*$–module* $\mathcal{M}_*$ consists of a family of graded right $\mathcal{R}_\alpha$–modules $\mathcal{M}_\alpha = \bigoplus_{j\in\mathbb{Z}} \mathcal{M}_\alpha^j$ together with a compatible collection of transition maps for $\alpha \subseteq \beta$ that are degree preserving homomorphisms of right $\mathcal{R}_\beta$–modules,

$$p_{\alpha\beta}^*(\mathcal{M}_\alpha) := p_{\alpha\beta}^{-1}(\mathcal{M}_\alpha) \otimes_{p_{\alpha\beta}^{-1}(\mathcal{R}_\alpha)} \mathcal{R}_\beta \to \mathcal{M}_\beta\,.$$

As we only consider commutative DG algebras, the sign rule allows to switch the module structure from right to left, in particular one can form the tensor product $\mathcal{M}_* \otimes_{\mathcal{R}_*} \mathcal{N}_*$ of $\mathcal{R}_*$–modules, defining it simplex by simplex through $(\mathcal{M}_* \otimes_{\mathcal{R}_*} \mathcal{N}_*)_\alpha = \mathcal{M}_\alpha \otimes_{\mathcal{R}_\alpha} \mathcal{N}_\alpha$.

If $\mathcal{M}_*$ is equipped with a differential $\partial = \partial_{\mathcal{M}_*}$ such that $(\mathcal{M}_*, \partial)$ becomes a DG module over $\mathcal{R}_*$ then we call $\mathcal{M}_*$ a DG $\mathcal{R}_*$–module in brief. Such a module is said to have *coherent cohomology* if $\mathcal{H}^i(\mathcal{M}_\alpha)$, the cohomology with respect to the differential on $\mathcal{M}_\alpha$, is a coherent $\mathcal{O}_{W_\alpha}$–module for each simplex α and each integer i; it is said to be *(locally) bounded above* if for each simplex $\mathcal{H}^i(\mathcal{M}_\alpha) = 0$ for $i \gg 0$, and it is said to *vanish (locally) above* if already $\mathcal{M}_\alpha^i = 0$ for $i \gg 0$.

2.4. Next we comment upon and fix some of the usual notations and conventions from homological algebra that extend in a straightforward manner to (DG) $\mathcal{R}_*$–modules. We include these details in order to be able later to calibrate Atiyah classes against Chern classes: Over time, the conventions and signs in constructing mapping cones, distinguished triangles and their associated extension classes have changed, say from [Har] over [Ver] to [ALG]; see also the comments in [SGA4, Exp. XVII] and [Del3]. Whereas classically, e.g. in [At,] or [Ill, V.5.4.1, 5.3.3], the first

Chern class is the trace of the opposite of the Atiyah class, it appears in [ALJ] as the trace of the Atiyah class itself.

If i is an integer, $\mathcal{M}_*[i]$ is the shifted module with $\mathcal{M}_*[i]^n = \mathcal{M}_*^{i+n}$ and the *same right $\mathcal{R}_*$–module structure*, whence the *left* structure becomes $f(m[i]) = (-1)^{|i||f|}(fm)[i]$. In case $\mathcal{M}_*$ is a DG module, $\mathcal{M}_*[i]$ becomes a DG module with respect to the differential $\partial_{\mathcal{M}_*[i]} = (-1)^i \partial_{\mathcal{M}_*}$. Writing the shift functor on the left, say as $T^i\mathcal{M}_* = \mathcal{M}_*[i]$, and considering T as an operator of degree -1, the conventions just introduced obey the usual sign rule.

A *morphism of $\mathcal{R}_*$–modules* $\mathcal{M}_* \to \mathcal{N}_*$ of degree i is a collection of homomorphisms $f_\alpha : \mathcal{M}_\alpha \to \mathcal{N}_\alpha$ of right $\mathcal{R}_\alpha$–modules that satisfy $f_\alpha(\mathcal{M}_\alpha^j) \subseteq \mathcal{M}_\alpha^{i+j}$ for all $j \in \mathbb{Z}$ and are compatible with the transition maps. By convention, if no degree is specified, a homomorphism of $\mathcal{R}_*$–modules is assumed to be of degree 0. We set

$$\mathrm{Hom}_{\mathcal{R}_*}(\mathcal{M}_*, \mathcal{N}_*) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}^i_{\mathcal{R}_*}(\mathcal{M}_*, \mathcal{N}_*)\,,$$

where $\mathrm{Hom}^i_{\mathcal{R}_*}(\mathcal{M}_*, \mathcal{N}_*)$ denotes the morphisms of degree i. If $\mathcal{M}_*$, $\mathcal{N}_*$ are DG modules with differential ∂ then $\mathrm{Hom}_{\mathcal{R}_*}(\mathcal{M}_*, \mathcal{N}_*)$ is a complex of vector spaces with differential $h \mapsto [\partial, h] := \partial h - (-1)^{|h|} h\partial$. Note that for any integer j the maps $h[j]$ and h are identical once the grading is ignored, and so one usually suppresses the shift in the notation of morphisms.

2.5. The morphisms of degree i are thus the (degree preserving) homomorphisms $\mathcal{M}_* \to \mathcal{N}_*[i]$ of $\mathcal{R}_*$–modules, the cycles of degree i in the Hom–complex are the homomorphisms $\mathcal{M}_* \to \mathcal{N}_*[i]$ of DG $\mathcal{R}_*$–modules, whereas the boundaries are the homomorphisms that are homotopic to zero. The *homotopy category* $K(\mathcal{R}_*)$ of DG $\mathcal{R}_*$–modules has as its objects the DG $\mathcal{R}_*$–modules but its morphisms are the homotopy classes, $\mathrm{Hom}_{K(\mathcal{R}_*)}(\mathcal{M}_*, \mathcal{N}_*) = \mathrm{H}^0(\mathrm{Hom}_{\mathcal{R}_*}(\mathcal{M}_*, \mathcal{N}_*))$. The *derived category* $D(\mathcal{R}_*)$ is obtained from $K(\mathcal{R}_*)$ as usual by inverting all quasiisomorphisms. Adornements such as $(\ )_c^{a,b}$ or $(\ )_c^b$ determine full subcategories of DG modules, where a bounds the underlying graded objects on each simplex, b bounds the cohomology, and c refers to special structure of the cohomology modules. For example, $D^-_{coh}(\mathcal{R}_*)$ denotes the derived category of DG $\mathcal{R}_*$–modules that are bounded above with coherent cohomology. In case $\mathcal{R}_*$ is a structure sheaf, such as $\mathcal{O}_{X_*}$, we write Hom_{X_*} and $D(X_*)$ to reduce clutter.

The morphisms in $D(\mathcal{R}_*)$ are denoted

$$\mathrm{Ext}^k_{\mathcal{R}_*}(\mathcal{M}_*, \mathcal{N}_*) := \mathrm{Hom}_{D(\mathcal{R}_*)}(\mathcal{M}_*, \mathcal{N}_*[k])\,, \quad k \in \mathbb{Z}\,.$$

Similarly, if X is a complex space and $\mathcal{M}$, $\mathcal{N}$ are complexes of $\mathcal{O}_X$–modules then $\mathrm{Ext}^k_X(\mathcal{M}, \mathcal{N})$ will represent the set of morphisms of degree k in the derived category $D(X)$. Note that if $\mathcal{N}$ is bounded below and so admits an injective resolution, say, $\mathcal{I}$ then these Ext-groups are given as usual by the cohomology of the complex $\mathrm{Hom}_X(\mathcal{M}, \mathcal{I})$. This definition allows in particular to define $H^k(X, \mathcal{N}) := \mathrm{Ext}^k_X(\mathcal{O}_X, \mathcal{N})$ for any complex $\mathcal{N}$ of $\mathcal{O}_X$–modules. Below we will show how to compute these groups using projective "resolutions" on X_*.

2.6. For a homomorphism $f : \mathcal{N}'_* \to \mathcal{N}_*$ of DG $\mathcal{R}_*$–modules, its *mapping cone*, $\mathrm{Con}_*(f) := \mathcal{N}_* \oplus \mathcal{N}'_*[1]$, is formed simplex by simplex according to the conventions of [ALG, X.36ff] so that for local sections n of $\mathcal{N}_\alpha$ and $Tn' = n'[1]$ of $\mathcal{N}'_\alpha[1]$ the differential in $\mathrm{Con}_*(f)$ maps (n, Tn') to $(\partial_{\mathcal{N}_*}(n) - f(n'), -T\partial_{\mathcal{N}'_*}(n'))$. The mapping

cone is again a DG $\mathcal{R}_*$–module, and the triangulated structure of either $K(\mathcal{R}_*)$ or $D(\mathcal{R}_*)$ is now defined in terms of the distinguished triangles arising from mapping cones.

2.7. Generalizing the construction of mapping cones, if $\mathcal{M}_*^\bullet \equiv (\cdots \to \mathcal{M}_*^{(i)} \xrightarrow{\delta^i} \mathcal{M}_*^{(i+1)} \to \cdots)$ is a complex of DG $\mathcal{R}_*$–modules, then the associated total complex $\bar{\mathcal{M}}_* = \prod_i \mathcal{M}_*^{(i)}[i]$ is a DG $\mathcal{R}_*$–module as well. The following simple fact will be used throughout: If the complex $\mathcal{M}_*^\bullet$ is acyclic and locally vanishes above, then the associated DG module $\bar{\mathcal{M}}_*$ is again acyclic.

2.8. As just done, we will often argue *simplex by simplex* and such arguments are alleviated by the following: For each simplex α, the restriction functor $\mathcal{M}_* \mapsto \mathcal{M}_\alpha$ is exact and admits a left adjoint p_α^* defined through

$$p_\alpha^*(\mathcal{M}_\alpha)_\beta := \begin{cases} p_{\alpha\beta}^*(\mathcal{M}_\alpha) & \text{for } \alpha \subseteq \beta \\ 0 & \text{otherwise,} \end{cases}$$

whence one has

$$\operatorname{Hom}_{\mathcal{R}_*}(p_\alpha^*(\mathcal{M}_\alpha), \mathcal{N}_*) \cong \operatorname{Hom}_{\mathcal{R}_\alpha}(\mathcal{M}_\alpha, \mathcal{N}_\alpha) \tag{2}$$

for each $\mathcal{R}_\alpha$–module $\mathcal{M}_\alpha$ and each $\mathcal{R}_*$–module $\mathcal{N}_*$, see [Fle1, §2], [BMi, 4.1], or, even more generally, [Ill, VI.5.3]. The functor p_α^* is compatible with differentials, it is as well a left adjoint to the restriction functor on the category of DG $\mathcal{R}_*$–modules.

2.9. Inductive arguments based on the dimension of simplices often use the *skeleton filtration* on an $\mathcal{R}_*$–module $\mathcal{M}_*$ given by

$$\mathcal{M}_{\le k} = \operatorname{Im}\Big(\bigoplus_{|\alpha|\le k} p_\alpha^*(\mathcal{M}_\alpha) \to \mathcal{M}_*\Big) \subseteq \mathcal{M}_*\,.$$

For a DG module this filtration is by DG submodules.

2.10. Analogous facts and notions allow to argue *degree by degree* on W_α for each individual simplex α. Consider, in general, (DG) $\mathcal{R}$–modules $\mathcal{M} = \bigoplus_{i\in\mathbb{Z}} \mathcal{M}^i$ over a (negatively graded DG) $\mathcal{O}_W$–algebra $\mathcal{R} = \bigoplus_{i\le 0} \mathcal{R}^i$ with $(W, \mathcal{O}_W)$ some ringed space. For each fixed degree $i \in \mathbb{Z}$, the restriction functor $\mathcal{M} \mapsto \mathcal{M}^i$ to $\mathcal{O}_W$–modules is exact and admits as left adjoint the functor $\mathcal{V} \mapsto \mathcal{V} \otimes_{\mathcal{O}_W} \mathcal{R}[-i]$. The corresponding canonical $\mathcal{R}$–homomorphism $\bigoplus_{i\in\mathbb{Z}} \mathcal{M}^i \otimes_{\mathcal{O}_W} \mathcal{R}[-i] \to \mathcal{M}$ is an epimorphism of (DG) $\mathcal{R}$–modules. The analogue to the skeleton filtration is the *degree filtration*

$$\mathcal{M}^{\ge k} = \operatorname{Im}\Big(\bigoplus_{i\ge k} \mathcal{M}^i \otimes_{\mathcal{O}_W} \mathcal{R}[-i] \to \mathcal{M}\Big) \subseteq \mathcal{M}\,,$$

whence $\mathcal{M}^{\ge k}$ is just the (DG) submodule of $\mathcal{M}$ generated by all homogeneous components in degrees at least k.

Now we turn to the existence and construction of resolutions for DG $\mathcal{R}_*$–modules.

2.11. We will say in brief that an $\mathcal{R}_*$–module $\mathcal{M}_* = \bigoplus_{i\in\mathbb{Z}} \mathcal{M}_*^i$ has *coherent homogeneous components* if each $\mathcal{M}_\alpha^i$ is coherent as $\mathcal{O}_{W_\alpha}$–module. *By convention, our DG algebras will always have coherent homogeneous components.* The category of all $\mathcal{R}_*$–modules with coherent homogeneous components is then abelian and will,

slightly abusively, be denoted $\mathbf{Coh}\,\mathcal{R}_*$. The restriction to a simplex α and its left adjoint p_α^* preserve coherence of homogeneous components and the $\mathcal{R}_*$–module $\bigoplus_{\alpha\in A} p_\alpha^*(\mathcal{M}_\alpha)$ is in $\mathbf{Coh}\,\mathcal{R}_*$ along with $\mathcal{M}_*$.

2.12. A module $\mathcal{P}_* \in \mathbf{Coh}\,\mathcal{R}_*$ will be called *projective* if it is a projective object in that category, equivalently, if the functor $\mathrm{Hom}_{\mathcal{R}_*}(\mathcal{P}_*, -)$ is exact on $\mathbf{Coh}\,\mathcal{R}_*$. As remarked above, such a projective module is in general not a projective object in the category of all $\mathcal{R}_*$–modules. Note also that, in general, $\mathcal{R}_*$ is not a projective module over itself. In contrast, $\mathcal{R}_\alpha$, being a module with coherent homogeneous components over the Stein compact set W_α, is projective as a module over itself, and if $\mathcal{P}_\alpha$ is projective in $\mathbf{Coh}\,\mathcal{R}_\alpha$ then $p_\alpha^*(\mathcal{P}_\alpha)$ is a projective $\mathcal{R}_*$–module since a left adjoint of an exact functor preserves projectives. A module $\mathcal{F}_* \in \mathbf{Coh}\,\mathcal{R}_*$ is called *graded free* if it is a direct sum of modules $p_\alpha^*(\mathcal{P}_\alpha)$ with graded free $\mathcal{R}_\alpha$–modules $\mathcal{P}_\alpha$. Thus graded free modules are always projective. Each category $\mathbf{Coh}\,\mathcal{R}_\alpha$ has enough projectives, for example, the graded free modules. As the canonical $\mathcal{R}_*$–homomorphism $\bigoplus_{\alpha\in A} p_\alpha^*(\mathcal{M}_\alpha) \to \mathcal{M}_*$ is an epimorphism, there are enough projectives in $\mathbf{Coh}\,\mathcal{R}_*$ as well.

Using the simplicial structure, and generalizing slightly [Fle1, 2.3, (3)], one obtains the following precise description of projective $\mathcal{R}_*$–modules.

Lemma 2.13. 1. *An $\mathcal{R}_*$–module $\mathcal{P}_*$ is projective if and only if*

$$\mathcal{P}_* \cong \bigoplus\nolimits_{\alpha\in A} p_\alpha^*(\mathcal{Q}_\alpha)$$

for some projective $\mathcal{R}_\alpha$–modules $\mathcal{Q}_\alpha$ with coherent homogeneous components.

2. *If $\mathcal{Q}_\alpha$ is a projective $\mathcal{R}_\alpha$–module with $\mathcal{Q}_\alpha^i = 0$ for $i \gg 0$, then $\mathcal{Q}_\alpha \cong \bigoplus_{j\in\mathbb{Z}} \mathcal{Q}_\alpha^{(j)}$, where $\mathcal{Q}_\alpha^{(j)}$ is a projective $\mathcal{R}_\alpha$–module generated in degree j.*

Proof. By induction on the dimension of simplices, it clearly suffices to show the following. If $A' \subseteq A$ consists of all simplices α' such that $\mathcal{P}_\alpha = 0$ for each strict subset $\alpha \subset \alpha'$, then $\bigoplus_{\alpha'\in A'} p_{\alpha'}^*(\mathcal{P}_{\alpha'})$ is a direct summand of $\mathcal{P}_*$. As $\mathcal{P}_*$ is projective the canonical epimorphism $\varphi : \bigoplus_{\alpha\in A} p_\alpha^*(\mathcal{P}_\alpha) \to \mathcal{P}_*$ admits a section ψ . The definition of A' yields for each $\alpha' \in A'$ that $\varphi_{\alpha'}$, and thus also $\psi_{\alpha'}$, is the identity on $(\bigoplus_{\alpha\in A} p_\alpha^*(\mathcal{P}_\alpha))_{\alpha'} = \mathcal{P}_{\alpha'}$. Composing the projection $\bigoplus_{\alpha\in A} p_\alpha^*(\mathcal{P}_\alpha) \to p_{\alpha'}^*(\mathcal{P}_{\alpha'})$ with ψ then retracts the natural homomorphism $p_{\alpha'}^*(\mathcal{P}_{\alpha'}) \to \mathcal{P}_*$ and the claim (1) follows.

Repeating the argument with respect to degrees instead of simplices yields the second assertion. □

If $\mathcal{P}_*$ is a projective $\mathcal{R}_*$–module which also carries a DG structure then we will say in brief that $\mathcal{P}_*$ is a projective DG $\mathcal{R}_*$–module, although, of course, $\mathcal{P}_*$ is not necessarily a projective object in the category of all DG modules with coherent homogeneous components.

For a projective module $\mathcal{P}_*$ the skeleton filtration 2.9 is by direct summands: In the notation of the preceding lemma, $\mathcal{P}_{\le k} \cong \bigoplus_{|\alpha|\le k} p_\alpha^*(\mathcal{Q}_\alpha)$, whence each such submodule as well as each subquotient $\mathcal{P}_{\le k}/\mathcal{P}_{\le k-1} \cong \bigoplus_{|\alpha|=k} p_\alpha^*(\mathcal{Q}_\alpha)$ is again projective. Analogously, if $\mathcal{P}_\alpha$ vanishes above, each submodule or subquotient with respect to the degree filtration is again projective. If $\mathcal{P}_*$, resp. $\mathcal{P}_\alpha$, is a projective DG module, then $\mathcal{P}_{\le k}$, resp. $\mathcal{P}_\alpha^{\ge k}$, is not necessarily a direct summand as DG module, but one has the following simple observations that reflect typical reduction steps when dealing with projective DG $\mathcal{P}_*$–modules that locally vanish above.

Lemma 2.14. *If $\mathcal{M}_*$ and $\mathcal{P}_*$ are DG $\mathcal{R}_*$–modules with $\mathcal{P}_*$ projective, then the natural map*

$$H^p(\mathrm{Hom}_{\mathcal{R}_*}(\mathcal{P}_*, \mathcal{M}_*)) \longrightarrow \varprojlim_k H^p(\mathrm{Hom}_{\mathcal{R}_*}(\mathcal{P}_{\leq k}, \mathcal{M}_*))$$

is bijective for each integer p.

If $\mathcal{M}_\alpha$ and $\mathcal{P}_\alpha$ are DG $\mathcal{R}_\alpha$–modules with $\mathcal{P}_\alpha$ projective and $\mathcal{P}^i_\alpha = 0$ for $i \gg 0$, then the natural map

$$H^p(\mathrm{Hom}_{\mathcal{R}_\alpha}(\mathcal{P}_\alpha, \mathcal{M}_\alpha)) \longrightarrow \varprojlim_k H^p(\mathrm{Hom}_{\mathcal{R}_\alpha}(\mathcal{P}^{\geq k}_\alpha, \mathcal{M}_\alpha))$$

is bijective for each integer p.

If $\mathcal{P}_\alpha$ is generated in a single degree k, then there is a natural number s such that the complex $\mathrm{Hom}_{\mathcal{R}_\alpha}(\mathcal{P}_\alpha, \mathcal{M}_\alpha)$ becomes a direct summand of $\Gamma(W_\alpha, \mathcal{M}_\alpha[k])^s$.

Proof. As said above, $0 \to \mathcal{P}_{\leq k} \to \mathcal{P}_* \to \mathcal{P}_*/\mathcal{P}_{\leq k} \to 0$ is an exact sequence of projective $\mathcal{R}_*$–modules and so splits, whence the induced map $\mathrm{Hom}_{\mathcal{R}_*}(\mathcal{P}_*, \mathcal{M}_*) \to \mathrm{Hom}_{\mathcal{R}_*}(\mathcal{P}_{\leq k}, \mathcal{M}_*)$ is surjective. The first claim follows immediately and the second one is obtained replacing skeleton by degree filtration.

For the final assertion, note that the hypothesis yields a natural epimorphism of DG $\mathcal{R}_\alpha$–modules $\mathcal{P}^k_\alpha \otimes_{\mathcal{O}_{W_\alpha}} \mathcal{R}_\alpha[-k] \to \mathcal{P}_\alpha$. As $\mathcal{P}_\alpha$ is projective it has in particular coherent homogeneous components and so $\mathcal{P}^k_\alpha$ can be generated by a finite number of sections over W_α, which in turn provides an epimorphism $\mathcal{R}_\alpha[-k]^s \to \mathcal{P}_\alpha$. Employing now fully that $\mathcal{P}_\alpha$ is projective, this epimorphism splits as a homomorphism of $\mathcal{R}_\alpha$–modules. As $\mathcal{P}_\alpha$ is generated in a single degree, the differential necessarily vanishes on the generators and the splitting is already a morphism of DG modules. Finally use that $\mathrm{Hom}_{\mathcal{R}_\alpha}(\mathcal{R}_\alpha[-k], \mathcal{M}_\alpha) = \Gamma(W_\alpha, \mathcal{M}_\alpha[k])$. □

An $\mathcal{R}_*$–module $\mathcal{M}_*$ will be called *W_*-acyclic* if $H^i(W_\alpha, \mathcal{M}_\alpha) = 0$ for all $\alpha \in A$. For instance, by Theorem B every module $\mathcal{M}_* \in \mathbf{Coh}\,\mathcal{R}_*$ is W_*-acyclic. The point of the definition is of course that a quasiisomorphism $\mathcal{M}_* \to \mathcal{N}_*$ of W_*-acyclic DG modules induces quasiisomorphisms $\Gamma(W_\alpha, \mathcal{M}_\alpha) \to \Gamma(W_\alpha, \mathcal{N}_\alpha)$ over each simplex.

Lemma 2.15. *For each DG $\mathcal{R}_*$–module $\mathcal{M}_*$ there is a W_*-acyclic resolution, thus a quasiisomorphism $\mathcal{M}_* \hookrightarrow \widetilde{\mathcal{W}}_*$ from $\mathcal{M}_*$ into a W_*-acyclic module $\widetilde{\mathcal{W}}_*$. The construction is functorial in $\mathcal{M}_*$.*

Proof. Consider on each simplex $\alpha \in A$ the canonical flabby resolution, say, $\mathcal{M}_\alpha \to \mathcal{W}^\bullet_\alpha$. As this resolution is functorial, each component of the complex $\mathcal{W}^\bullet_\alpha$ is naturally again a DG $\mathcal{R}_\alpha$–module and the resulting system $\mathcal{M}_* \to \mathcal{W}^\bullet_*$ is a resolution by DG $\mathcal{R}_*$–modules. Now cut this resolution on the simplex α at the place $d(\alpha) := \sum_{\beta \subseteq \alpha} \dim W_\beta$ to obtain $\bar{\mathcal{W}}^i_\alpha := \mathcal{W}^i_\alpha$ for $i < d(\alpha)$, $\bar{\mathcal{W}}^i_\alpha := 0$ for $i > d(\alpha)$ and $\bar{\mathcal{W}}^i_\alpha := \mathrm{Ker}(\mathcal{W}^i_\alpha \to \mathcal{W}^{i+1}_\alpha)$ for $i = d(\alpha)$.

The choice of the cut-off point guarantees first, as $d(\alpha) \geq \dim W_\alpha$, that $\bar{\mathcal{W}}^{d(\alpha)}_\alpha$ is W_*-acyclic along with the other components. Secondly, as $d(\alpha) \leq d(\beta)$ for $\alpha \subseteq \beta$, the truncated complexes still form a simplicial system. Finally, $\bar{\mathcal{W}}^\bullet_*$ is now a complex that locally vanishes above and thus the canonical injection $\mathcal{M}_* \hookrightarrow \widetilde{\mathcal{W}}_*$ into the DG module obtained as the total complex associated to $\bar{\mathcal{W}}^\bullet_*$ provides the desired functorial quasiisomorphism into a W_*-acyclic $\mathcal{R}_*$–module. □

Corollary 2.16. *The inclusion of the full subcategory of $D(X_*)$ consisting of all W_*-acyclic DG $\mathcal{R}_*$–modules into $D(X_*)$ is an equivalence of triangulated categories. An inverse is given by associating to a DG $\mathcal{R}_*$–module its W_*-acyclic resolution.*

To derive a functor on all DG $\mathcal{R}_*$–modules it suffices thus to derive it on the W_*-acyclic ones. □

Proposition 2.17. *Let $\mathcal{M}_* \to \mathcal{N}_*$ be a quasiisomorphism of DG $\mathcal{R}_*$–modules and $\mathcal{P}_*$ a projective DG $\mathcal{R}_*$–module that locally vanishes above.*

1. *The map $\mathcal{P}_* \otimes_{\mathcal{R}_*} \mathcal{M}_* \to \mathcal{P}_* \otimes_{\mathcal{R}_*} \mathcal{N}_*$ is a quasiisomorphism.*
2. *If $\mathcal{M}_*, \mathcal{N}_*$ are W_*-acyclic, then $\mathrm{Hom}_{\mathcal{R}_*}(\mathcal{P}_*, \mathcal{M}_*) \longrightarrow \mathrm{Hom}_{\mathcal{R}_*}(\mathcal{P}_*, \mathcal{N}_*)$ is a quasiisomorphism. In particular, any morphism $\mathcal{P}_* \to \mathcal{N}_*$ of DG $\mathcal{R}_*$–modules lifts through the given quasiisomorphism to a morphism of DG $\mathcal{R}_*$–modules $\mathcal{P}_* \to \mathcal{M}_*$.*

Proof. Assertion (1) can be verified locally and there [ALG, X.66,§4 no.3] applies.

To prove (2), we may assume by the first part of lemma 2.14 that $\mathcal{P}_* = \mathcal{P}_{\leq k}$. In this case the spectral sequence associated to the skeleton filtration on $\mathcal{P}_*$ converges and so it is sufficient to show the claim in case that $\mathcal{P}_* \cong p^*(\mathcal{P}_\alpha)$ with a projective DG $\mathcal{R}_\alpha$–module $\mathcal{P}_\alpha$. In view of 2.8(2) this requires to show that the corresponding map

$$(*) \qquad \mathrm{Hom}_{\mathcal{R}_\alpha}(\mathcal{P}_\alpha, \mathcal{M}_\alpha) \to \mathrm{Hom}_{\mathcal{R}_\alpha}(\mathcal{P}_\alpha, \mathcal{N}_\alpha).$$

is a quasiisomorphism. Because of the second part of lemma 2.14 we may reduce to the situation where $\mathcal{P}_\alpha$ is generated in finitely many degrees, in which case the spectral sequence associated to the degree filtration on $\mathcal{P}_\alpha$ converges. It remains to deal with the case that $\mathcal{P}_\alpha$ is generated in a single degree k and then the final part of lemma 2.14 exhibits the map $(*)$ as a direct summand of $\Gamma(W_\alpha, \mathcal{M}_\alpha[k]^s) \to \Gamma(W_\alpha, \mathcal{N}_\alpha[k]^s)$ for some s. As $\mathcal{M}_*, \mathcal{N}_*$ are W_*-acyclic, this last map is a quasiisomorphism and the claim follows. □

Corollary 2.18. *Any quasiisomorphism $\mathcal{P}_* \to \mathcal{Q}_*$ between projective DG $\mathcal{R}_*$–modules that locally vanish above is a homotopy equivalence. In particular, for every DG $\mathcal{R}_*$–module $\mathcal{M}_*$ and any quasiisomorphism $\mathcal{P}_* \to \mathcal{Q}_*$ the induced maps*

$$\mathrm{Hom}_{\mathcal{R}_*}(\mathcal{Q}_*, \mathcal{M}_*) \longrightarrow \mathrm{Hom}_{\mathcal{R}_*}(\mathcal{P}_*, \mathcal{M}_*) \qquad \textit{and}$$
$$\mathcal{P}_* \otimes_{\mathcal{R}_*} \mathcal{M}_* \longrightarrow \mathcal{Q}_* \otimes_{\mathcal{R}_*} \mathcal{M}_*$$

are quasiisomorphisms.

Proof. It follows from 2.16 and 2.17(2) that any quasiisomorphism between projective DG $\mathcal{R}_*$–modules that locally vanish above is represented by an actual morphism $f : \mathcal{P}_* \to \mathcal{Q}_*$ of DG $\mathcal{R}_*$–modules. Such a morphism is a homotopy equivalence if the identity on the mapping cone $\mathcal{C}_* = \mathrm{Con}_*(f)$ is homotopic to 0. With $\mathcal{P}_*, \mathcal{Q}_*$ also $\mathcal{C}_*$ is projective, thus W_*-acyclic, and locally vanishes above. So the preceding result 2.17(2) applies to yield that $\mathrm{Hom}_{\mathcal{R}_*}(\mathcal{C}_*, \mathcal{C}_*) \to \mathrm{Hom}_{\mathcal{R}_*}(\mathcal{C}_*, 0) = 0$ is a quasiisomorphism. The rest follows immediately. □

A quasiisomorphism $h : \mathcal{P}_* \to \mathcal{M}_*$ of DG $\mathcal{R}_*$–modules will be called a *projective approximation* of $\mathcal{M}_*$ if $\mathcal{P}_*$ is a projective DG $\mathcal{R}_*$–module that locally vanishes above. The preceding corollary shows that projective approximations of a DG $\mathcal{R}_*$–module are unique up to homotopy equivalence. Their existence is settled next.

Proposition 2.19. *A DG $\mathcal{R}_*$–module $\mathcal{M}_*$ admits a projective approximation if and only if it is bounded above with coherent cohomology.*

Proof. If $h : \mathcal{P}_* \to \mathcal{M}_*$ is a projective approximation, then $\mathcal{M}_*$ is necessarily bounded above with coherent cohomology as this holds for $\mathcal{P}_*$ and h is a quasiisomorphism. Conversely, it suffices by 2.15 to show the existence of a projective approximation when $\mathcal{M}_*$ is furthermore W_*-acyclic. But if $\mathcal{M}_*$ is W_*-acyclic with coherent, thus W_*-acyclic, cohomology, then its submodule of boundaries, $\mathcal{B}_* = \partial(\mathcal{M}_*) \subseteq \mathcal{M}_*$ is also W_*-acyclic: the exact sequences

$$0 \to \mathcal{K}^i_* \to \mathcal{M}^i_* \to \mathcal{B}^i_* \to 0$$
$$0 \to \mathcal{B}^{i-1}_* \to \mathcal{K}^i_* \to \mathcal{H}^i(\mathcal{M}_*) \to 0$$

yield for each $j \geq 1$ and each simplex $\alpha \in A$ isomorphisms

$$H^j(W_\alpha, \mathcal{B}^i_\alpha) \xrightarrow{\cong} H^{j+1}(W_\alpha, \mathcal{K}^i_\alpha) \xleftarrow{\cong} H^{j+1}(W_\alpha, \mathcal{B}^{i-1}_\alpha),$$

whence $H^j(W_\alpha, \mathcal{B}^i_\alpha) \cong H^{j+\dim W_\alpha}(W_\alpha, \mathcal{B}^{i-\dim W_\alpha}_\alpha) = 0$ for each i.

In view of this observation, the classical construction of a projective resolution applies: as $\mathcal{H}_*(\mathcal{M}_*)$ is coherent there exists a surjection from a graded free $\mathcal{R}^0_*$–module $\mathcal{Q}_*$ with coherent homogeneous components onto $\mathcal{H}(\mathcal{M}_*)$ such that $\mathcal{Q}^i_\alpha$ vanishes in degrees $i > d$ if $\mathcal{H}^i(\mathcal{M}_\beta)$ vanishes for those degrees for all simplices $\beta \subseteq \alpha$. Due to the acyclicity of the boundaries $\mathcal{B}_* \subseteq \mathcal{M}_*$, this homomorphism lifts from the cohomology to the cycles $\mathcal{K}_* \subseteq \mathcal{M}_*$, and the resulting morphism of DG $\mathcal{R}_*$–modules $\mathcal{P}^{(0)}_* := \mathcal{Q}_* \otimes_{\mathcal{R}^0_*} \mathcal{R}_* \to \mathcal{M}_*$ is surjective in cohomology. Now $\mathcal{P}^{(0)}_*$ is projective, thus any submodule is W_*-acyclic, and the usual process, see e.g. [Har, I.4.6], produces a complex $\cdots \to \mathcal{P}^{(i)}_* \to \cdots \to \mathcal{P}^{(0)}_* \to \mathcal{M}_* \to 0$ with projective DG $\mathcal{R}_*$–modules $\mathcal{P}^{(i)}_*$ that becomes acyclic when $\mathcal{H}$ is applied. Moreover, on each simplex α the projective modules $\mathcal{P}^{(i)}_\alpha$ can be choosen to vanish uniformly above so that the DG $\mathcal{R}_*$–module $\mathcal{P}_*$ associated to the total complex of the $\mathcal{P}^{(i)}_*$ is again a projective $\mathcal{R}_*$–module. The induced morphism $h : \mathcal{P}_* \to \mathcal{M}_*$ then resolves, cf. 2.7, and constitutes a desired projective approximation. □

Example 2.20. As we mentioned earlier in 2.12, and as can be seen easily from 2.13, $\mathcal{R}_*$ is in general not a projective module over itself. Instead, $\mathcal{R}_*$ admits the following explicit, and natural, projective approximation. First consider the co-Čech complex

$$\cdots \to \bigoplus_{|\alpha|=p} p^*_\alpha(\mathcal{R}_\alpha)e(\alpha) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \bigoplus_{|\alpha|=1} p^*_\alpha(\mathcal{R}_\alpha)e(\alpha) \xrightarrow{\delta} \bigoplus_{|\alpha|=0} p^*_\alpha(\mathcal{R}_\alpha)e(\alpha) \to 0\,,$$

where the direct sums are indexed by ordered simplices, $\alpha = (\alpha_0, \ldots, \hat{\alpha}_j, \ldots, \alpha_p)$, and the differential δ is dual to the differential of the Čech complex so that

$$\delta(e(\alpha_0, \ldots, \alpha_p)) = \sum (-1)^j e(\alpha_0, \ldots, \hat{\alpha}_j, \ldots, \alpha_p)\,.$$

This differential is a homomorphism of DG modules and each term of the complex is a projective $\mathcal{R}_*$–module by 2.13. The augmentation map $\bigoplus_{|\alpha|=0} p^*_\alpha(\mathcal{R}_\alpha)e(\alpha) \xrightarrow{\epsilon} \mathcal{R}_*$, sum of the natural maps $p^*_\alpha(\mathcal{R}_\alpha) \to \mathcal{R}_*$, realizes the co-Čech complex as a projective resolution of $\mathcal{R}_*$ as an $\mathcal{R}_*$–module: the restriction of the augmented complex to any simplex is evidently contractible.

The total complex associated to this co-Čech complex gives thus a projective approximation $\mathcal{P}_*$ of $\mathcal{R}_*$, see 2.7 or the proof of 2.19. Note that $\mathcal{P}_* = \bigoplus p^*_\alpha(\mathcal{R}_\alpha)[|\alpha|]$, where the sum is taken over all ordered simplices α.

Remarks 2.21. 1. It follows that a projective approximation $h : \mathcal{P}_* \to \mathcal{M}_*$ can be realized as a pair of morphisms of DG $\mathcal{R}_*$–modules $\mathcal{P}_* \to \widetilde{\mathcal{M}}_* \leftarrow \mathcal{M}_*$, where the morphism $\mathcal{M}_* \to \widetilde{\mathcal{M}}_*$ is a W_*-acyclic resolution and $\mathcal{P}_* \to \widetilde{\mathcal{M}}_*$ is a projective resolution as constructed in the proof.

2. If so desired, one may clearly modify the above construction to obtain in the end a projective approximation $\mathcal{F}_* \to \mathcal{M}_*$ with $\mathcal{F}_*$ a *graded free* DG $\mathcal{R}_*$–module.

3. With the notation of the proposition, if $A' \subseteq A$ is a set of simplices such that $\mathcal{H}(\mathcal{M}_\alpha) = 0$ for all simplices α that do not contain a simplex from A', then the above construction produces a projective approximation with $\mathcal{P}_\alpha = 0$ for the same simplices.

Analogously, if $\mathcal{H}^i(\mathcal{M}_*) = 0$ for $i > i_0$, then the construction provides a projective approximation $h : \mathcal{P}_* \to \mathcal{M}_*$ with $\mathcal{P}^i_* = 0$ for $i > i_0$.

2.22. Now we can describe the Ext-functors of DG $\mathcal{R}_*$–modules when the contravariant argument admits a projective approximation. Recall that by our conventions for $\mathcal{R}_*$–modules $\mathcal{M}_*, \mathcal{N}_*$ the Ext^k–functors are given by the set of morphisms $\mathcal{M}_* \to \mathcal{N}_*$ of degree k in the derived category so that

$$\mathrm{Ext}^k_{\mathcal{R}_*}(\mathcal{M}_*, \mathcal{N}_*) = \mathrm{Hom}_{D(\mathcal{R}_*)}(\mathcal{M}_*, \mathcal{N}_*[k]) .$$

The following proposition summarizes 2.19, 2.17(2), and 2.16. The reader should compare the result with [Ill, VI.10.2.4] in the algebraic case.

Proposition 2.23. *If $\mathcal{M}_*$ is a DG $\mathcal{R}_*$–modules that is bounded above with coherent cohomology, and $\mathcal{N}_*$ is any DG $\mathcal{R}_*$–module then*

$$\mathrm{Ext}^k_{\mathcal{R}_*}(\mathcal{M}_*, \mathcal{N}_*) \cong H^k(\mathrm{Hom}_{\mathcal{R}_*}(\mathcal{P}_*, \tilde{\mathcal{N}}_*)) \quad \textit{for} \quad k \in \mathbb{Z} ,$$

where $\mathcal{P}_* \to \mathcal{M}_*$ is a projective approximation and $\mathcal{N}_* \to \tilde{\mathcal{N}}_*$ is a quasiisomorphism into a W_*-acyclic DG $\mathcal{R}_*$–module. □

2.24. Projective approximations allow as well to derive tensor products of DG $\mathcal{R}_*$–modules. Calling a DG $\mathcal{R}_*$–module $\mathcal{M}_*$ *flat* if $\mathcal{M}_* \otimes_{\mathcal{R}_*} (\ \ )$ preserves quasiisomorphisms of DG $\mathcal{R}_*$–modules, a projective module that locally vanishes above is flat by 2.17(1). Indeed, for flatness it suffices already that the module locally vanishes above and that its restriction to each simplex is projective. For example, $\mathcal{R}_*$ itself is always flat. If $\mathcal{P}_* \to \mathcal{M}_*$ is a projective approximation of a DG $\mathcal{R}_*$–module $\mathcal{M}_*$, necessarily bounded above with coherent cohomology, and if $\mathcal{N}_*$ is any DG $\mathcal{R}_*$–module, then $\mathcal{P}_* \otimes_{\mathcal{R}_*} \mathcal{N}_*$ represents $\mathcal{M}_* \underline{\otimes}_{\mathcal{R}_*} \mathcal{N}_*$, the *derived tensor product* of $\mathcal{M}_*$ with $\mathcal{N}_*$ over $\mathcal{R}_*$.

By 2.18, the derived tensor product is well defined up to homotopy equivalence, and by 2.17(1) and 2.18 a pair of quasiisomorphisms $\mathcal{M}_* \to \mathcal{M}'_*, \mathcal{N}_* \to \mathcal{N}'_*$ of DG $\mathcal{R}_*$–modules induces a quasiisomorphism $\mathcal{M}_* \underline{\otimes}_{\mathcal{R}_*} \mathcal{N}_* \longrightarrow \mathcal{M}'_* \underline{\otimes}_{\mathcal{R}_*} \mathcal{N}'_*$. If $\mathcal{N}_*$ admits a projective approximation as well, say $\mathcal{Q}_* \to \mathcal{N}_*$, then $\mathcal{M}_* \otimes_{\mathcal{R}_*} \mathcal{Q}_*$ represents $\mathcal{M}_* \underline{\otimes}_{\mathcal{R}_*} \mathcal{N}_*$ too. If $\mathcal{N}_*$ is flat and locally vanishes above then, again by [ALG, X.Prop.4], the given quasiisomorphism $\mathcal{P}_* \to \mathcal{M}_*$ induces a quasiisomorphism $\mathcal{M}_* \underline{\otimes}_{\mathcal{R}_*} \mathcal{N}_* \longrightarrow \mathcal{M}_* \otimes_{\mathcal{R}_*} \mathcal{N}_*$ and so, for example, $\mathcal{M}_* \underline{\otimes}_{\mathcal{R}_*} \mathcal{R}_* \cong \mathcal{M}_*$.

A useful consequence of the preceding considerations is the following result that is again well known in the algebraic case, see [Ill, VI.10.3.15].

Corollary 2.25. *Let $\mathcal{R}_* \to \mathcal{S}_*$ be a quasiisomorphism of DG algebras over $\mathcal{O}_{W_*}$.*

1. *If $\mathcal{M}_*$, $\mathcal{N}_*$ are DG $\mathcal{R}_*$–modules that are bounded above with coherent cohomology, then there are natural isomorphisms*
$$\mathrm{Ext}^k_{\mathcal{R}_*}(\mathcal{M}_*, \mathcal{N}_*) \xrightarrow{\cong} \mathrm{Ext}^k_{\mathcal{S}_*}(\mathcal{M}_* \underline{\otimes}_{\mathcal{R}_*} \mathcal{S}_*, \mathcal{N}_* \underline{\otimes}_{\mathcal{R}_*} \mathcal{S}_*)$$
for each integer k.
2. *If $\mathcal{M}_*$ is in $D^-_{coh}(\mathcal{R}_*)$ and $\mathcal{N}_*$ any DG $\mathcal{S}_*$–module, then for each integer k there is a natural isomorphism*
$$\mathrm{Ext}^k_{\mathcal{R}_*}(\mathcal{M}_*, \mathcal{N}_*) \xrightarrow{\cong} \mathrm{Ext}^k_{\mathcal{S}_*}(\mathcal{M}_* \underline{\otimes}_{\mathcal{R}_*} \mathcal{S}_*, \mathcal{N}_*)\,.$$
3. *Restriction of scalars from $\mathcal{S}_*$ to $\mathcal{R}_*$ and $(\ \ )\underline{\otimes}_{\mathcal{R}_*} \mathcal{S}_*$ form a pair of inverse exact equivalences between $D^-_{coh}(\mathcal{S}_*)$ and $D^-_{coh}(\mathcal{R}_*)$.*

Proof. Let $\mathcal{P}_* \to \mathcal{M}_*$ and $\mathcal{Q}_* \to \mathcal{N}_*$ be projective approximations as DG $\mathcal{R}_*$–modules. By 2.17(1), $\mathcal{Q}_* \to \mathcal{Q}_* \otimes_{\mathcal{R}_*} \mathcal{S}_*$ is a quasiisomorphism and so
$$\mathrm{Ext}^k_{\mathcal{R}_*}(\mathcal{P}_*, \mathcal{Q}_*) \xrightarrow{\cong} \mathrm{Ext}^k_{\mathcal{R}_*}(\mathcal{P}_*, \mathcal{Q}_* \otimes_{\mathcal{R}_*} \mathcal{S}_*)\,.$$
As $\mathrm{Hom}_{\mathcal{R}_*}(\mathcal{P}_*, \mathcal{Q}_* \otimes_{\mathcal{R}_*} \mathcal{S}_*) \cong \mathrm{Hom}_{\mathcal{S}_*}(\mathcal{P}_* \otimes_{\mathcal{R}_*} \mathcal{S}_*, \mathcal{Q}_* \otimes_{\mathcal{R}_*} \mathcal{S}_*)$ and $\mathcal{P}_* \otimes_{\mathcal{R}_*} \mathcal{S}_*$ is a projective DG $\mathcal{S}_*$–module that locally vanishes above, assertion (1) follows from 2.23.

To obtain (2), let $\mathcal{P}_* \to \mathcal{M}_*$ be as before and $\mathcal{N}_* \to \widetilde{\mathcal{N}}_*$ a W_*–acyclic resolution that is a morphism of DG $\mathcal{S}_*$–modules. One has then
$$\begin{aligned}
\mathrm{Ext}^k_{\mathcal{R}_*}(\mathcal{M}_*, \mathcal{N}_*) &\cong H^k(\mathrm{Hom}_{\mathcal{R}_*}(\mathcal{P}_*, \widetilde{\mathcal{N}}_*)) && \text{by 2.23,}\\
&\cong H^k(\mathrm{Hom}_{\mathcal{S}_*}(\mathcal{P}_* \otimes_{\mathcal{R}_*} \mathcal{S}_*, \widetilde{\mathcal{N}}_*)) && \text{by adjunction,}\\
&\cong \mathrm{Ext}^k_{\mathcal{S}_*}(\mathcal{M}_* \underline{\otimes}_{\mathcal{R}_*} \mathcal{S}_*, \mathcal{N}_*) && \text{by 2.23 again.}
\end{aligned}$$
For (3), note that $(\ \ )\underline{\otimes}_{\mathcal{R}_*} \mathcal{S}_*$ on $D^-_{coh}(\mathcal{R}_*)$ is fully faithful by (1). This functor takes its values in $D^-_{coh}(\mathcal{S}_*)$, and to establish it is an equivalence with inverse as indicated, it suffices to remark that for each $\mathcal{N}_*$ in $D^-_{coh}(\mathcal{S}_*)$ the natural morphism $\mathcal{N}_* \underline{\otimes}_{\mathcal{R}_*} \mathcal{S}_* \to \mathcal{N}_*$ obtained from (2) is an isomorphism in $D^-_{coh}(\mathcal{S}_*)$. Indeed, let $\mathcal{N}_* \to \widetilde{\mathcal{N}}_*$ be again a W_*–acyclic resolution that is a morphism of DG $\mathcal{S}_*$–modules and choose a morphism of DG $\mathcal{R}_*$–modules $\mathcal{Q}_* \to \widetilde{\mathcal{N}}_*$ that constitutes a projective approximation. As $\widetilde{\mathcal{N}}_*$ is already a $\mathcal{S}_*$–module, this quasiisomorphism factors as $\mathcal{Q}_* \to \mathcal{Q}_* \otimes_{\mathcal{R}_*} \mathcal{S}_* \to \widetilde{\mathcal{N}}_*$, the morphism $\mathcal{Q}_* \to \mathcal{Q}_* \otimes_{\mathcal{R}_*} \mathcal{S}_*$ is a quasiisomorphism by 2.17(1), and thus so is $\mathcal{Q}_* \otimes_{\mathcal{R}_*} \mathcal{S}_* \to \widetilde{\mathcal{N}}_*$. Now the pair of quasiisomorphisms $\mathcal{Q}_* \otimes_{\mathcal{R}_*} \mathcal{S}_* \to \widetilde{\mathcal{N}}_* \leftarrow \mathcal{N}_*$ represents the morphism $\mathcal{N}_* \underline{\otimes}_{\mathcal{R}_*} \mathcal{S}_* \to \mathcal{N}_*$, whence the latter is an isomorphism in $D^-_{coh}(\mathcal{S}_*)$. □

2.26. Considering $\mathcal{O}_{X_*}$ as a DG algebra concentrated in degree 0, (projective) DG $\mathcal{O}_{X_*}$–modules are just complexes of (projective) $\mathcal{O}_{X_*}$–modules. In this situation, the restriction functor to a simplex α is easily seen to admit a right adjoint, given by
$$p_{\alpha *}(\mathcal{M}_\alpha)_\beta := \begin{cases} p_{\beta\alpha *}(\mathcal{M}_\alpha) & \text{for } \beta \subseteq \alpha \\ 0 & \text{otherwise,} \end{cases}$$

cf. [Fle1, §2], or, again more generally, [Ill, VI.5.3]. As it is right adjoint to an exact functor, $p_{\alpha *}$ transforms an injective $\mathcal{O}_{X_\alpha}$–module $\mathcal{I}_\alpha$ into the injective $\mathcal{O}_{X_*}$–module $p_{\alpha *}(\mathcal{I}_\alpha)$. The canonical map $\mathcal{M}_* \hookrightarrow \prod_{\alpha \in A} p_{\alpha *}(\mathcal{M}_\alpha)$ is a monomorphism for each $\mathcal{O}_{X_*}$–module $\mathcal{M}_*$, and embedding in turn each $\mathcal{M}_\alpha$ into an injective $\mathcal{O}_{X_\alpha}$–module $\mathcal{I}_\alpha$ yields by composition a monomorphism $\mathcal{M}_* \hookrightarrow \prod_{\alpha \in A} p_{\alpha *}(\mathcal{I}_\alpha)$ into an injective $\mathcal{O}_{X_*}$–module. Thus the category of $\mathcal{O}_{X_*}$–modules has enough injectives. In particular, for complexes $\mathcal{M}_*, \mathcal{N}_*$ of $\mathcal{O}_{X_*}$–modules one may calculate $\operatorname{Ext}^k_{X_*}(\mathcal{M}_*, \mathcal{N}_*)$ in the "classical" way as $\mathrm{H}^k(\operatorname{Hom}_{X_*}(\mathcal{M}_*, \mathcal{I}_*))$ if $\mathcal{N}_*$ admits an injective resolution $\mathcal{N}_* \to \mathcal{I}_*$. By 2.23, if $\mathcal{M}_*$ is a complex of $\mathcal{O}_{X_*}$–modules that is bounded above with coherent cohomology, then these groups can be calculated using a projective approximation of $\mathcal{M}_*$.

2.27. Restricting a given $\mathcal{O}_X$–module $\mathcal{M}$ to the Stein compact sets of the given covering defines the $\mathcal{O}_{X_*}$–module $\mathcal{M}_* = j^*\mathcal{M}$ with $\mathcal{M}_\alpha := \mathcal{M}|X_\alpha$. This functor is exact and so induces directly a functor $j^* : D(X) \to D(X_*)$ between the derived categories.

To describe a right adjoint, denote $j_\alpha : X_\alpha \hookrightarrow X$ the inclusion and associate to a module $\mathcal{M}_*$ on X_* the *Čech complex* $C^\bullet(\mathcal{M}_*)$ with terms

$$C^p(\mathcal{M}_*) := \prod_{|\alpha|=p} j_{\alpha *}(\mathcal{M}_\alpha)\,,$$

where the product is over all ordered simplices, and differential defined in the usual way by means of the transition maps for $\mathcal{M}_*$ and the given ordering on the simplices. The functor $j_*(\mathcal{M}_*) := \mathcal{H}^0(C^\bullet(\mathcal{M}_*))$ is a right adjoint to j^* on the category of $\mathcal{O}_{X_*}$–modules, and the canonical homomorphism of $\mathcal{O}_X$–modules $\mathcal{M} \to j_*j^*(\mathcal{M})$ is an isomorphism.

As the given covering is locally finite, the complex $C^\bullet(\mathcal{M}_*)$ is locally bounded; the localization at a point $x \in X$ vanishes in degrees greater than $\max\{|\alpha|, x \in X_\alpha\}$. As the covering is by closed sets, the functors $j_{\alpha *}$, and then also $C^p(\ \ )$, are exact. These two facts together imply that the total complex (associated to) $C^\bullet(\mathcal{M}^\bullet_*)$ is acyclic whenever $\mathcal{M}^\bullet_*$ is an acyclic complex of $\mathcal{O}_{X_*}$–modules. Accordingly, $C^\bullet$ can be viewed as a functor from $D(X_*)$ to $D(X)$. Note that the terms of the complex $C^\bullet(\mathcal{M}^\bullet_*)$ are flat $\mathcal{O}_X$–modules whenever $\mathcal{M}^\bullet_\alpha$ is flat over $\mathcal{O}_{X_\alpha}$ for each simplex α.

We now show that $C^\bullet$ represents Rj_*, the right derived functor of j_*: for an $\mathcal{O}_{X_*}$–module of the form $p_{\alpha *}(\mathcal{M}_\alpha)$, the complex $C^\bullet(p_{\alpha *}(\mathcal{M}_\alpha))$ is nothing but the usual (sheafified) Čech complex of $\mathcal{M}_\alpha$ on X_α with respect to the trivial covering $\{X_\alpha \cap X_i = X_\alpha\}_{i \in \alpha}$, then extended by zero to the rest of X. Clearly $C^\bullet(p_{\alpha *}(\mathcal{M}_\alpha))$ resolves $j_*(p_{\alpha *}(\mathcal{M}_\alpha)) \cong j_{\alpha *}(\mathcal{M}_\alpha)$, the extension of $\mathcal{M}_\alpha$ by zero. By 2.26, each $\mathcal{O}_{X_*}$–module $\mathcal{M}_*$ admits a resolution by $\mathcal{O}_{X_*}$–modules of the form $\prod_{\alpha \in A} p_{\alpha *}(\mathcal{N}_\alpha)$, whence the natural morphism of functors $Rj_* \to C^\bullet$, induced by the universal property of the derived functor, is indeed an isomorphism. The relationship between $D(X)$ and $D(X_*)$ can now be summarized as follows.

Proposition 2.28. *The functor $j^* : D(X) \to D(X_*)$ embeds $D(X)$ as a full and exact subcategory into $D(X_*)$ and $C^\bullet \cong Rj_*$ is an exact right adjoint. In particular,*

for $\mathcal{M},\mathcal{N}\in D(X)$ and $\mathcal{N}'_*\in D(X_*)$ there are functorial isomorphisms

$$\mathcal{M}\cong C^\bullet(j^*\mathcal{M})\,,$$

$$\operatorname{Ext}^k_X(\mathcal{M},\mathcal{N})\cong \operatorname{Ext}^k_{X_*}(j^*(\mathcal{M}),j^*(\mathcal{N}))\,,\quad \textit{and}$$

$$\operatorname{Ext}^k_{X_*}(j^*(\mathcal{M}),\mathcal{N}'_*)\cong \operatorname{Ext}^k_X(\mathcal{M},C^\bullet(\mathcal{N}'_*))\,.$$

The adjoint pair j^*, Rj_* satisfies the projection formula: *the canonical morphism*

$$\mathcal{M}\,\underline{\otimes}_{\mathcal{O}_X} Rj_*(\mathcal{N}_*)\to Rj_*(j^*\mathcal{M}\,\underline{\otimes}_{\mathcal{O}_{X_*}}\mathcal{N}_*)$$

is an isomorphism in $D(X)$ for $\mathcal{M}\in D(X)$ and $\mathcal{N}_*\in D^-_{coh}(X_*)$.

Proof. As mentioned above, the natural morphism of functors $\mathrm{id}\to j_*j^*$ is an isomorphism on the level of modules. It induces an isomorphism $\mathrm{id}\to (Rj_*)j^*\cong C^\bullet j^*$ of the corresponding derived functors on $D(X)$, whence the functor j^* is still fully faithful on $D(X)$.

To prove the projection formula, observe first that the natural map

$$j^*(\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{M}')\to j^*(\mathcal{M})\otimes_{\mathcal{O}_{X_*}} j^*(\mathcal{M}')$$

is an isomorphism as this holds on each simplex. As j^* is exact, this isomorphism passes to the derived tensor products as soon as those exist. Adjunction then yields a morphism $\mathcal{M}\,\underline{\otimes}_{\mathcal{O}_X}\mathcal{M}'\to Rj_*(j^*(\mathcal{M})\,\underline{\otimes}_{\mathcal{O}_{X_*}} j^*(\mathcal{M}'))$. Now set $\mathcal{M}'=Rj_*\mathcal{N}_*$ and compose the corresponding morphism with the one induced by the adjunction map $j^*Rj_*(\mathcal{N}_*)\to\mathcal{N}_*$ to obtain the morphism in the projection formula. Given the existence of this natural morphism, to establish it as an isomorphism for $\mathcal{N}_*\in D^-_{coh}(X_*)$, we first replace $\mathcal{N}_*$ by a projective approximation $\mathcal{P}_*$, then use 2.13 to reduce to the case $\mathcal{N}_*=p^*_\alpha(\mathcal{O}_{X_\alpha})$ for some simplex α. Now $Rj_*(p^*_\alpha(\mathcal{O}_{X_\alpha}))\cong C^\bullet(\mathcal{O}_X|X_\alpha)$ is a finite complex of flat $\mathcal{O}_X$–modules that resolves $\mathcal{O}_X|X_\alpha$ and $Rj_*(j^*\mathcal{M}\,\underline{\otimes}_{\mathcal{O}_{X_*}} p^*_\alpha(\mathcal{O}_{X_\alpha}))\cong C^\bullet(\mathcal{M}|X_\alpha)$ resolves $\mathcal{M}|X_\alpha$. The desired isomorphism in $D(X)$ follows thus from the obvious one for $\mathcal{O}_X$–modules, $\mathcal{M}|X_\alpha\cong\mathcal{M}\otimes_{\mathcal{O}_X}(\mathcal{O}_X|X_\alpha)$. □

Example 2.29. If $\mathcal{N}$ is any complex on X then, by our conventions, its cohomology groups $H^k(X,\mathcal{N})$ are the groups $\operatorname{Ext}^k_X(\mathcal{O}_X,\mathcal{N})$ and so can be computed by the complex $\operatorname{Hom}_{X_*}(\mathcal{P}_*,\tilde{\mathcal{N}}_*)$, where $\mathcal{P}_*\to\mathcal{O}_{X_*}$ is a projective resolution of $\mathcal{O}_{X_*}$ as a module over itself and $\mathcal{N}_*\to\tilde{\mathcal{N}}_*$ is a quasiisomorphism into a W_*-acyclic complex. Using the projective approximation $\mathcal{P}_*$ of $\mathcal{O}_{X_*}$ constructed in 2.20 it follows that $\operatorname{Hom}_{X_*}(\mathcal{P}_*,\tilde{\mathcal{N}}_*)$ is the usual Čech-complex $\Gamma(X,C^\bullet(\tilde{\mathcal{N}}_*))$.

Remark 2.30. 1. Note that the preceding constructions work more generally on the category of all sheaves of abelian groups $\mathbf{Ab}_X$ on X and $\mathbf{Ab}_{X_*}$ on X_*. In particular, later on we will need the Čech functor in this context, where it is still exact and defines with the same arguments as before the derived functor Rj_* from $D(\mathbf{Ab}_{X_*})$ into $D(\mathbf{Ab}_X)$.

2. As j^* is fully faithful, the essential image of $D(X)$ under this functor is easily seen to consist of all complexes $\mathcal{N}_*$ in $D(X_*)$ for which the transition maps $p^*_{\alpha\beta}(\mathcal{N}_\alpha)\to\mathcal{N}_\beta$ are quasiisomorphisms for all simplices α,β with $\alpha\subseteq\beta$. In this case the natural map $j^*C^\bullet(\mathcal{N}_*)\to\mathcal{N}_*$ is a quasiisomorphism, and 2.28 implies that for each complex $\mathcal{M}\in D(X)$, there are isomorphisms

$$\operatorname{Ext}^i_{X_*}(\mathcal{N}_*,j^*\mathcal{M})\cong\operatorname{Ext}^i_X(C^\bullet(\mathcal{N}_*),\mathcal{M})\,,\quad i\in\mathbb{Z}\,.$$

We now turn to the construction of algebra resolutions and resolvents.

2.31. A morphism $\mathcal{R}_* \to \widetilde{\mathcal{R}}_*$ of DG algebras over W_* is called *free* if there is a graded free DG $\mathcal{R}_*$–module $\mathcal{F}_*$ such that $\widetilde{\mathcal{R}}_* \cong \mathbb{S}_{\mathcal{R}_*}(\mathcal{F}_*)$ as $\mathcal{R}_*$-algebras, where the symmetric algebra functor is to be understood in the graded context. Note that the composition of free morphisms of DG algebras is again a free morphism.

If $\mathcal{R}_* \to \mathcal{S}_*$ is any morphism of DG algebras over W_*, a factorization $\mathcal{R}_* \to \widetilde{\mathcal{R}}_* \to \mathcal{S}_*$ with $\mathcal{R}_* \to \widetilde{\mathcal{R}}_*$ free and $\widetilde{\mathcal{R}}_* \to \mathcal{S}_*$ a surjective quasiisomorphism of DG algebras will be called a DG *algebra resolution* of $\mathcal{S}_*$ over $\mathcal{R}_*$.

The result 2.25 together with the next one are the crucial ingredients in the construction of cotangent complexes following Quillen's original approach [Qui1], [Qui2]. In his framework of closed model categories, the free morphisms are the cofibrations and the surjective quasiisomorphisms are the acyclic fibrations.

Proposition 2.32. *Every morphism $\mathcal{R}_* \to \mathcal{S}_*$ of DG algebras admits a DG algebra resolution.*

Proof. According to our general assumption on DG algebras, the $\mathcal{R}_*$–module $\mathcal{S}_*$ is in $\mathbf{Coh}\,\mathcal{R}_*$, and so 2.19 and 2.21(a),(b) guarantee a projective approximation of $\mathcal{S}_*$ in form of a morphism of DG $\mathcal{R}_*$–modules from a graded free DG $\mathcal{R}_*$–module, say $\mathcal{F}_*^{(0)} \to \mathcal{S}_*$. The induced morphism $\mathcal{R}_*^{(0)} := \mathbb{S}_{\mathcal{R}_*}(\mathcal{F}_*^{(0)}) \to \mathcal{S}_*$ of DG algebras induces a surjection in cohomology and the structure map $\mathcal{R}_* \to \mathcal{R}_*^{(0)}$ is free.

Now assume constructed for some integer $k \geq 0$ a morphism of DG $\mathcal{R}_*$–algebras $\mathcal{R}_*^{(k)} \to \mathcal{S}_*$ with $\mathcal{R}_*^{(k)}$ free over $\mathcal{R}_*$ that is surjective in cohomology and such that $\mathcal{H}^i(\mathcal{R}_*^{(k)}) \to \mathcal{H}^i(\mathcal{S}_*)$ is an isomorphism for $i > -k$. As the kernel of the surjection $\mathcal{H}^{-k}(\mathcal{R}_*^{(k)}) \to \mathcal{H}^{-k}(\mathcal{S}_*)$ is coherent, one may choose a graded free coherent $\mathcal{O}_{W_*}$–module $\mathcal{F}_*^{-k-1}$ that is concentrated in degree $-k-1$ and a morphism $\varphi : \mathcal{F}_*^{-k-1} \to \mathcal{R}_*^{(k)}$ of graded $\mathcal{O}_{W_*}$–modules of degree 1 that maps $\mathcal{F}_*^{-k-1}$ into the cycles of $\mathcal{R}_*^{(k)}$ and such that the sequence of $\mathcal{O}_{W_*}$–modules

$$\mathcal{F}_*^{-k-1} \longrightarrow \mathcal{H}^{-k}(\mathcal{R}_*^{(k)}) \longrightarrow \mathcal{H}^{-k}(\mathcal{S}_*) \longrightarrow 0$$

is exact. Now set

$$\mathcal{R}_*^{(k+1)} \cong \mathbb{S}_{\mathcal{R}_*^{(k)}}(\mathcal{F}_*^{-k-1} \otimes_{\mathcal{O}_{W_*}} \mathcal{R}_*^{(k)}) \cong \mathbb{S}_{\mathcal{R}_*}(\mathcal{F}_*^{-k-1} \otimes_{\mathcal{O}_{W_*}} \mathcal{R}_*) \otimes_{\mathcal{R}_*} \mathcal{R}_*^{(k)}$$

and use φ to extend the given differential. This DG algebra is free over $\mathcal{R}_*^{(k)}$ and the structure map is an isomorphism in degrees greater than $-(k+1)$. The composed map, say, $\gamma : \mathcal{F}_*^{-k-1} \to \mathcal{R}_*^{(k)} \to \mathcal{S}_*$ maps $\mathcal{F}_*^{-k-1}$ into the boundaries, hence we can find a lifting $\tilde{\gamma} : \mathcal{F}_*^{-k-1} \to \mathcal{S}_*^{-k-1}$ so that $\gamma = \partial\tilde{\gamma}$. There is a unique homorphism of $\mathcal{R}_*^{(k)}$-algebras $\mathcal{R}_*^{(k+1)} \to \mathcal{S}_*$ that restricts to $\tilde{\gamma}$ on $\mathcal{F}_*^{-k-1}$. By construction it is also a morphism of DG algebras, and it is surjective in cohomology with $\mathcal{H}^i(\mathcal{R}_*^{(k+1)}) \to \mathcal{H}^i(\mathcal{S}_*)$ an isomorphism for each $i > -(k+1)$. Finally set $\widetilde{\mathcal{R}}_* = \varinjlim_k \mathcal{R}_*^{(k)}$. $\square$

To remember the graded free DG $\mathcal{R}_*$–modules $\mathcal{F}_*^{-k} \otimes_{\mathcal{O}_{W_*}} \mathcal{R}_*$ that were successively adjoined in the construction of the algebra resolution we also write $\widetilde{\mathcal{R}}_* = \mathcal{R}_*[\mathcal{F}_*^{-k} \otimes_{\mathcal{O}_{W_*}} \mathcal{R}_*; k \geq 0]$, and the sequence of free DG algebra morphisms

$$\mathcal{R}_* \hookrightarrow \cdots \hookrightarrow \widetilde{\mathcal{R}}_*^{(k)} := \mathcal{R}_*[\mathcal{F}_*^{-i} \otimes_{\mathcal{O}_{W_*}} \mathcal{R}_*; k \geq i \geq 0] \hookrightarrow \cdots \hookrightarrow \widetilde{\mathcal{R}}_*$$

is sometimes called the associated *Postnikov tower*.

2.33. Following [Pal, Fle1] we will introduce the notion of a resolvent. Given a morphism of complex spaces $X \to Y$, a *resolvent* for X over Y consists in a triple $(X_*, W_*, \mathcal{R}_*)$ satisfying the following conditions.

1. X_* is the simplicial space associated to some locally finite covering $(X_i)_{i\in I}$ of X by Stein compact subsets, see 2.2;
2. $X_* \hookrightarrow W_*$ is a smoothing of $X \to Y$ as in 2.2;
3. $\mathcal{R}_*$ is a free DG algebra resolution of $\mathcal{O}_{X_*}$ over $\mathcal{O}_{W_*}$.

Given X_* and W_*, we will sometimes also refer to $\mathcal{R}_*$ as a resolvent of $\mathcal{O}_{X*}$. The preceding results have the following application.

Corollary 2.34. *A resolvent $(X_*, W_*, \mathcal{R}_*)$ of X over Y exists and one may assume that $\mathcal{O}_{W_*} \to \mathcal{R}^0_*$ is an isomorphism, thus that $\mathcal{R}_* = \mathcal{O}_{W_*}[\mathcal{F}^{-k}_*; k \geq 1]$ with suitable graded free $\mathcal{O}_{W_*}$–modules $\mathcal{F}^{-k}_*$ concentrated in degree $-k$.*

The induced functor $(\ \ )\underline{\otimes}_{\mathcal{R}_*} \mathcal{O}_{X_*} : D^-_{coh}(\mathcal{R}_*) \to D^-_{coh}(X_*)$ is an exact equivalence of categories. □

Remark 2.35. Up to this point, all results are valid with insignificant modifications for a finitely presented morphism $X \to Y$ of arbitrary locally noetherian schemes if one replaces "Stein compact set" by "affine scheme". If one further replaces "coherent" by "quasi-coherent", the preceding results hold even for any morphism of schemes.

We finish this section with the relevant results on cotangent complexes, and here we need characteristic zero, as otherwise DG algebra resolutions are not sufficient, [Qui2], [Qui3].

2.36. Let $\mathcal{R}_*$ be a DG algebra over W_* as before. By definition, $\mathcal{O}_{W_\alpha \times_Y W_\alpha} \cong \mathcal{O}_{W_\alpha} \tilde{\otimes}_{\mathcal{O}_Y} \mathcal{O}_{W_\alpha}$, where $\tilde{\otimes}$ denotes the *analytic tensor product*. Abusively, we set

$$\mathcal{R}_* \otimes_{\mathcal{O}_Y} \mathcal{R}_* := \mathcal{R}_* \otimes_{\mathcal{O}_{W_*}} (\mathcal{O}_{W_*} \tilde{\otimes}_{\mathcal{O}_Y} \mathcal{O}_{W_*}) \otimes_{\mathcal{O}_{W_*}} \mathcal{R}_*\,,$$

and note that $\mathcal{R}_* \otimes_{\mathcal{O}_Y} \mathcal{R}_*$ is naturally a DG algebra over the smooth simplical scheme $W_* \times_Y W_* = \{W_\alpha \times_Y W_\alpha\}_{\alpha \in A}$ of Stein compact sets, see 2.2.

Let $\mu : \mathcal{R}_* \otimes_{\mathcal{O}_Y} \mathcal{R}_* \to \mathcal{R}_*$ denote the multiplication map and set $\mathcal{I}_* := \ker \mu \subseteq \mathcal{R}_* \otimes_{\mathcal{O}_Y} \mathcal{R}_*$. The DG $\mathcal{R}_*$–module $\Omega^1_{\mathcal{R}_*/Y} = \mathcal{I}_*/\mathcal{I}^2_*$ is the *module of (analytic) differential 1-forms* of $\mathcal{R}_*$ over Y. As $\mathcal{R}_*$ has coherent homogeneous components by hypothesis, $\Omega^1_{\mathcal{R}_*/Y}$ is a DG $\mathcal{R}_*$–module in $\mathbf{Coh}(\mathcal{R}_*)$. The *universal derivation* $d : \mathcal{R}_* \to \Omega^1_{\mathcal{R}_*/Y}$ maps a local section f of $\mathcal{R}_\alpha$ to the class

$$df = 1 \otimes f - f \otimes 1 \in \mathcal{I}_\alpha \bmod \mathcal{I}^2_\alpha\,.$$

It is a map of degree zero that has the desired universal property with respect to homogeneous Y-derivations into graded $\mathcal{R}_*$–modules in $\mathbf{Coh}\,\mathcal{R}_*$. More precisely, with $\mathrm{Der}^i_Y(\mathcal{R}_*, \mathcal{N}_*)$ the group of Y-derivations of degree i into the $\mathcal{R}_*$–module $\mathcal{N}_*$, one has a natural inclusion

$$\mathrm{Hom}^i_{\mathcal{R}_*}(\Omega^1_{\mathcal{R}_*/Y}, \mathcal{N}_*) \xhookrightarrow{(\ \ )\circ d} \mathrm{Der}^i_Y(\mathcal{R}_*, \mathcal{N}_*)$$

that becomes an isomorphism for $\mathcal{N}_* \in \mathbf{Coh}\,\mathcal{R}_*$. As the classes df locally generate $\Omega^1_{\mathcal{R}_*/Y}$, its differential, inherited from $\mathcal{R}_* \otimes_{\mathcal{O}_Y} \mathcal{R}_*$, is uniquely determined through

$$\partial(df) = d(\partial f)\,.$$

The module of (analytic) differential forms of degree $k \geq 0$ is the DG $\mathcal{R}_*$–module $\Omega^k_{\mathcal{R}_*/Y} := \Lambda^k_{\mathcal{R}_*}\Omega^1_{\mathcal{R}_*/Y}$, the *alternating* or *(graded) exterior power* of $\Omega^1_{\mathcal{R}_*/Y}$, see, for example, [Lod, 5.4.3]. These DG modules are again in $\mathbf{Coh}\,\mathcal{R}_*$. For later use we recall that (graded) symmetric and exterior powers are related through $\mathbb{S}^k_{\mathcal{R}_*}(\Omega^1_{\mathcal{R}_*/Y}[1]) \cong \Omega^k_{\mathcal{R}_*/Y}[k]$.

2.37. The differential module $\Omega^1_{\mathcal{R}_*/Y}$ of a free algebra over $\mathcal{O}_{W_*}$ is a graded free $\mathcal{R}_*$–module and so are the exterior powers $\Omega^k_{\mathcal{R}_*/Y}$. The natural maps

$$\Omega^k_{\mathcal{R}_*/Y} \longrightarrow \Omega^k_{\mathcal{R}_*/Y} \otimes_{\mathcal{R}_*} \mathcal{O}_{X_*} \quad \text{for } k \geq 0$$

are thus quasiisomorphisms by 2.15. The Čech-complex $C^\bullet(\Omega^1_{\mathcal{R}_*/Y} \otimes_{\mathcal{R}_*} \mathcal{O}_{X_*})$ on X is a *cotangent complex* of X over Y and is denoted $\mathbb{L}_{X/Y}$. By construction, it is a complex in $D^-_{coh}(X)$ whose terms are flat $\mathcal{O}_X$–modules. The isomorphism class of $\mathbb{L}_{X/Y}$ in $D(X)$ is well defined, see [Fle1], in the sense that it does not depend on the choice of the resolvent $(X_*, W_*, \mathcal{R}_*)$. We set furthermore

$$\Lambda^k \mathbb{L}_{X/Y} := C^\bullet(\Omega^k_{\mathcal{R}_*/Y} \otimes_{\mathcal{R}_*} \mathcal{O}_{X_*}) \quad \text{for } k \geq 0\,.$$

These complexes are also in $D^-_{coh}(X)$ and their isomorphism classes are again well defined. Indeed, they represent the *derived exterior powers* of $\mathbb{L}_{X/Y}$ in the sense of [DPu, Qui2] or [Ill, I.4.2.2].

The functoriality of the formation of the cotangent complex and its powers has the following consequence: the natural morphisms of complexes of $\mathcal{O}_{X_\alpha}$–modules

$$\Lambda^k \mathbb{L}_{X/Y}\big|_{X_\alpha} \to \Omega^k_{\mathcal{R}_\alpha/Y} \otimes_{\mathcal{R}_\alpha} \mathcal{O}_{X_\alpha} \tag{3}$$

are quasiisomorphisms for each simplex α and each $k \geq 0$.

Recall that the tangent cohomology functors of X over Y are defined by

$$T^i_{X/Y}(\mathcal{N}) := \operatorname{Ext}^i_X(\mathbb{L}_{X/Y}, \mathcal{N})\,, \quad i \in \mathbb{Z}\,,$$

for any $\mathcal{O}_X$–module, or, more generally, for any complex $\mathcal{N}$ in $D(X)$. The following more explicit description in terms of resolvents will be frequently used in this paper.

Proposition 2.38. *If $(X_*, W_*, \mathcal{R}_*)$ is a resolvent of X over Y then for every complex $\mathcal{N} \in D(X)$ and each integer i there are canonical isomorphisms*

$$T^i_{X/Y}(\mathcal{N}) \cong \operatorname{Ext}^i_{\mathcal{R}_*}(\Omega^1_{\mathcal{R}_*/Y}, \mathcal{N}_*) \cong H^i(\operatorname{Hom}_{\mathcal{R}_*}(\Omega^1_{\mathcal{R}_*/Y}, \tilde{\mathcal{N}}_*))\,,$$

where $\mathcal{N}_* \to \tilde{\mathcal{N}}_*$ is a quasiisomorphism of $\mathcal{N}_* := j^*(\mathcal{N})$ into a W_*-acyclic complex of $\mathcal{O}_{X_*}$–modules. Moreover, if $\mathcal{N}$ is a complex of coherent $\mathcal{O}_X$–modules then these groups are as well isomorphic to $H^i(\operatorname{Der}_Y(\mathcal{R}_*, \mathcal{N}_*))$.

Proof. In view of 2.30(2), the quasiisomorphism (3) for $k = 1$ shows that

$$\operatorname{Ext}^i_{\mathcal{O}_{X_*}}(\Omega^1_{\mathcal{R}_*/Y} \otimes_{\mathcal{R}_*} \mathcal{O}_{X_*}, \mathcal{N}_*) \cong \operatorname{Ext}^i_X(\mathbb{L}_{X/Y}, \mathcal{N})\,,$$

and the term on the right is isomorphic to $T^i_{X/Y}(\mathcal{N})$. According to 2.25 (3), the term on the left is isomorphic to $\operatorname{Ext}^i_{\mathcal{R}_*}(\Omega^1_{\mathcal{R}_*/Y}, \mathcal{N}_*)$, and so the first isomorphism follows. The second one follows from the fact that $\Omega^1_{\mathcal{R}_*/Y}$ is a projective $\mathcal{R}_*$–module. The final assertion is a consequence of the usual universal property of the module of analytic differentials, see 2.36. $\square$

3. The Atiyah class

We first define and investigate Atiyah classes and their powers on $D^-_{coh}(\mathcal{R}_*)$ for any DG algebra $\mathcal{R}_*$ over W_* in terms of connections, then descend to $D^-_{coh}(X)$ by means of the Čech functor. We keep the notations of the preceding section, and unadorned tensor products will be over $\mathcal{R}_*$. Moreover ∂ will indiscriminately denote the differentials of the respective DG modules.

Atiyah classes via connections. Recall that a *connection* on an $\mathcal{R}_*$–module $\mathcal{M}_*$ is a map of degree 0,

$$\nabla : \mathcal{M}_* \longrightarrow \mathcal{M}_* \otimes \Omega^1_{\mathcal{R}_*/Y}\,,$$

that satisfies the usual product rule, $\nabla(mf) = \nabla(m)f + m \otimes df$, for local sections m in $\mathcal{M}_\alpha$ and f in $\mathcal{R}_\alpha$.

We first collect some basic and simple facts about connections.

Lemma 3.1. *Every projective $\mathcal{R}_*$–module that locally vanishes above admits a connection.*

Proof. As the direct sum of a family of connections is again a connection, we may restrict by 2.12 to the case that $\mathcal{P}_* \cong p^*_\alpha(\mathcal{P}_\alpha)$, where $\mathcal{P}_\alpha$ is a projective $\mathcal{R}_\alpha$–module generated in a single degree, say k. If $\mathcal{P}_\alpha$ is graded free, then $\mathcal{P}_\alpha \cong V[k] \otimes_{\mathbb{C}} \mathcal{R}_\alpha$ with V a finite dimensional vector space over $\mathbb{C}$, and the collection of maps

$$1 \otimes d : V[k] \otimes_{\mathbb{C}} \mathcal{R}_\beta \to V[k] \otimes_{\mathbb{C}} \Omega^1_{\mathcal{R}_\beta/Y}\,, \quad \alpha \subseteq \beta\,,$$

defines a connection on $V[k] \otimes_{\mathbb{C}} p^*_\alpha(\mathcal{R}_\alpha)$. In the general case $\mathcal{P}_\alpha$ embeds into a free module such that $\mathcal{F}_\alpha \cong V[k] \otimes_{\mathbb{C}} \mathcal{R}_\alpha \cong \mathcal{P}_\alpha \oplus \mathcal{Q}_\alpha$. If $\nabla : p^*_\alpha(\mathcal{F}_\alpha) \to p^*_\alpha(\mathcal{F}_\alpha) \otimes \Omega^1_{\mathcal{R}_*/Y}$ is a connection on $p^*_\alpha(\mathcal{F}_\alpha)$, then the composition

$$p^*_\alpha(\mathcal{P}_\alpha) \xrightarrow{incl.} p^*_\alpha(\mathcal{F}_\alpha) \xrightarrow{\nabla} p^*_\alpha(\mathcal{F}_\alpha) \otimes \Omega^1_{\mathcal{R}_*/Y} \xrightarrow{proj.} p^*_\alpha(\mathcal{P}_\alpha) \otimes \Omega^1_{\mathcal{R}_*/Y}$$

is easily seen to be a connection on $p^*_\alpha(\mathcal{P}_\alpha)$. □

Proposition 3.2. *For any connection $\nabla : \mathcal{M}_* \to \mathcal{M}_* \otimes \Omega^1_{\mathcal{R}_*/Y}$ on a DG $\mathcal{R}_*$–module $\mathcal{M}_*$, the map $[\partial, \nabla]$ of degree 1 is a homomorphism of DG $\mathcal{R}_*$–modules, so that*

$$[\partial, \nabla] = \partial\nabla - \nabla\partial \in \mathrm{Hom}^1_{\mathcal{R}_*}(\mathcal{M}_*, \mathcal{M}_* \otimes \Omega^1_{\mathcal{R}_*/Y})$$

is a cycle. Its cohomology class in $\mathrm{H}^1(\mathrm{Hom}_{\mathcal{R}_*}(\mathcal{M}_*, \mathcal{M}_* \otimes \Omega^1_{\mathcal{R}_*/Y}))$ is independent of the choice of connection.

Proof. That $[\partial, \nabla]$ is a homomorphism of right $\mathcal{R}_*$–modules is easily verified by explicit calculation. Moreover,

$$\big[\partial, [\partial, \nabla]\big] = \partial[\partial, \nabla] + [\partial, \nabla]\partial = \partial(\partial\nabla - \nabla\partial) + (\partial\nabla - \nabla\partial)\partial = \partial^2\nabla - \nabla\partial^2 = 0\,,$$

whence $[\partial, \nabla]$ is a homomorphism of DG modules, thus a cycle.

If $\nabla_1, \nabla_2 : \mathcal{M}_* \to \mathcal{M}_* \otimes \Omega^1_{\mathcal{R}_*/Y}$ are connections, then $\nabla_1 - \nabla_2$ is $\mathcal{R}_*$–linear and so $[\partial, \nabla_1] = [\partial, \nabla_2] + [\partial, \nabla_1 - \nabla_2]$, which means that the cycles $[\partial, \nabla_1], [\partial, \nabla_2]$ are cohomologous. □

This construction of a well defined cycle from a connection can be iterated: for each $k \geq 0$, the cycle $[\partial, \nabla]$ defines a morphism of DG $\mathcal{R}_*$–modules of degree k,

$$[\partial, \nabla]^k : \mathcal{M}_* \longrightarrow \mathcal{M}_* \otimes (\Omega^1_{\mathcal{R}_*/Y})^{\otimes k} \xrightarrow{1 \otimes \wedge^k} \mathcal{M}_* \otimes \Omega^k_{\mathcal{R}_*/Y}\,,$$

where $\wedge^k : (\Omega^1_{\mathcal{R}_*/Y})^{\otimes k} \to \Omega^k_{\mathcal{R}_*/Y}$ is the natural projection. The class of $[\partial, \nabla]^k$ in $\mathrm{H}^k(\mathrm{Hom}_{\mathcal{R}_*}(\mathcal{M}_*, \mathcal{M}_* \otimes \Omega^k_{\mathcal{R}_*/Y}))$ is again independent of the chosen connection as follows from the case $f = \mathrm{id}_{\mathcal{M}_*}$ in the next result that deals more generally with the functoriality of these iterated classes.

Lemma 3.3. *Let $f : \mathcal{M}_* \to \mathcal{M}'_*$ be a morphism of DG $\mathcal{R}_*$–modules. If*

$$\nabla : \mathcal{M}_* \to \mathcal{M}_* \otimes \Omega^1_{\mathcal{R}_*/Y} \quad \text{and} \quad \nabla' : \mathcal{M}'_* \to \mathcal{M}'_* \otimes \Omega^1_{\mathcal{R}_*/Y}$$

are connections, then $(f \otimes 1) \circ [\partial, \nabla]^k$ and $[\partial, \nabla']^k \circ f$ represent the same class in $\mathrm{H}^k(\mathrm{Hom}_{\mathcal{R}_*}(\mathcal{M}_*, \mathcal{M}'_* \otimes \Omega^k_{\mathcal{R}_*/Y}))$.

Proof. If $k = 0$, the classes in question are equal to f itself. If $k > 0$, the map

$$g := (f \otimes 1) \circ \nabla \circ [\partial, \nabla]^{k-1} - \nabla' \circ [\partial, \nabla']^{k-1} \circ f : \mathcal{M}_* \to \mathcal{M}'_* \otimes \Omega^k_{\mathcal{R}_*/Y}$$

of degree $k - 1$ is $\mathcal{R}_*$-linear. As $[\partial, g]$ is equal to $(f \otimes 1) \circ [\partial, \nabla]^k - [\partial, \nabla']^k \circ f$ the claim follows. □

Applying the preceding result when f is a homotopy equivalence between projective approximations of the same DG $\mathcal{R}_*$–module shows that the following definition is independent of the choice of projective approximation or connection on it.

Definition 3.4. Let $\mathcal{M}_*$ be a DG $\mathcal{R}_*$–module that is bounded above with coherent cohomology. Let $\mathcal{P}_* \to \mathcal{M}_*$ be a projective approximation as in 2.19 and let ∇ be a connection on $\mathcal{P}_*$ that exists by 3.1. The *Atiyah class* of $\mathcal{M}_*$ with respect to $\mathcal{R}_*/Y$ is the image of $[\partial, \nabla]$ under the isomorphism

$$\begin{array}{ccc} \mathrm{H}^1(\mathrm{Hom}_{\mathcal{R}_*}(\mathcal{P}_*, \mathcal{P}_* \otimes \Omega^1_{\mathcal{R}_*/Y})) & \cong & \mathrm{Ext}^1_{\mathcal{R}_*}(\mathcal{M}_*, \mathcal{M}_* \underline{\otimes}\, \Omega^1_{\mathcal{R}_*/Y}) \\ {[\partial, \nabla]} & \mapsto & \mathrm{At}(\mathcal{M}_*)\,, \end{array}$$

and the class of $[\partial, \nabla]^k$ is mapped to the *k-th power of the Atiyah class* of $\mathcal{M}_*$,

$$[\partial, \nabla]^k \mapsto \mathrm{At}^k(\mathcal{M}_*) \in \mathrm{Ext}^k_{\mathcal{R}_*}(\mathcal{M}_*, \mathcal{M}_* \underline{\otimes}\, \Omega^k_{\mathcal{R}_*/Y})\,.$$

3.5. Now let $\varphi : \mathcal{R}_* \to \mathcal{S}_*$ be a morphism of DG algebras over W_* and $\mathcal{M}_*$ an $\mathcal{R}_*$–module with connection ∇. The composition

$$\mathcal{M}_* \xrightarrow{\nabla \otimes 1} \mathcal{M}_* \otimes \Omega^1_{\mathcal{R}_*/Y} \otimes \mathcal{S}_* \xrightarrow{1 \otimes d\varphi} \mathcal{M}_* \otimes \Omega^1_{\mathcal{S}_*/Y} \xrightarrow{\cong} (\mathcal{M}_* \otimes \mathcal{S}_*) \otimes_{\mathcal{S}_*} \Omega^1_{\mathcal{S}_*/Y}$$

extends by the product rule to a connection $\nabla_{\mathcal{S}_*}$ on the $\mathcal{S}_*$–module $\mathcal{M}_* \otimes \mathcal{S}_*$. If $\mathcal{M}_*$ is a DG $\mathcal{R}_*$–module, one verifies easily that

$$[\partial, \nabla_{\mathcal{S}_*}]^k = (1 \otimes \wedge^k d\varphi) \circ ([\partial, \nabla]^k \otimes \mathcal{S}_*)\,,$$

where $\wedge^k d\varphi : (\Lambda^k_{\mathcal{R}_*} \Omega^1_{\mathcal{R}_*/Y}) \otimes \mathcal{S}_* \cong \Lambda^k_{\mathcal{S}_*}(\Omega^1_{\mathcal{R}_*/Y} \otimes \mathcal{S}_*) \to \Lambda^k_{\mathcal{S}_*} \Omega^1_{\mathcal{S}_*/Y}$ is the morphism of DG $\mathcal{S}_*$–modules induced by $d\varphi$. These considerations imply the following result.

Proposition 3.6. *Let $\varphi : \mathcal{R}_* \to \mathcal{S}_*$ be a morphism of DG algebras over W_* and $\mathcal{M}_* \in D^-_{coh}(\mathcal{R}_*)$. Under the natural maps induced by $d\varphi$,*

$$(\wedge^k d\varphi)_* : \mathrm{Ext}^k_{\mathcal{R}_*}(\mathcal{M}_*, \mathcal{M}_* \underline{\otimes}\, \Omega^k_{\mathcal{R}_*/Y}) \longrightarrow \mathrm{Ext}^k_{\mathcal{S}_*}(\mathcal{M}_* \underline{\otimes}\, \mathcal{S}_*, \mathcal{M}_* \underline{\otimes}\, \Omega^k_{\mathcal{S}_*/Y})\,,$$

the powers of the Atiyah class of $\mathcal{M}_*$ are mapped to those of $\mathcal{M}_* \underline{\otimes}\, \mathcal{S}_*$. □

If φ is a quasiisomorphism, then $(\wedge^k d\varphi)_*$ is an isomorphism for each k and in this sense the exact equivalence of triangulated categories $(\quad) \underline{\otimes}\, \mathcal{S}_* : D^-_{coh}(\mathcal{R}_*) \to D^-_{coh}(\mathcal{S}_*)$ from 2.25.3 commutes with powers of Atiyah classes.

3.7. The definition of a connection does not involve the grading of the underlying $\mathcal{R}_*$–module. In particular, if ∇ is a connection on the DG $\mathcal{R}_*$–module $\mathcal{M}_*$, it is as well one on the shifted module $\mathcal{M}_*[i]$ for each integer i. As $\partial_{\mathcal{M}_*[i]} = (-1)^i \partial_{\mathcal{M}_*}$, the canonical identification

$$\operatorname{Ext}^k_{\mathcal{R}_*}(\mathcal{M}_*, \mathcal{M}_* \underline{\otimes} \Omega^k_{\mathcal{R}_*/Y}) \xrightarrow[\cong]{[i]} \operatorname{Ext}^k_{\mathcal{R}_*}(\mathcal{M}_*[i], \mathcal{M}_*[i] \underline{\otimes} \Omega^k_{\mathcal{R}_*/Y})$$

for $\mathcal{M}_*$ in $D^-_{coh}(\mathcal{R}_*)$ maps $\operatorname{At}^k(\mathcal{M}_*)$ to $(-1)^{ki} \operatorname{At}^k(\mathcal{M}_*[i])$, thus, in short,

$$\operatorname{At}^k(\mathcal{M}_*[i]) = (-1)^{ki} \operatorname{At}^k(\mathcal{M}_*)[i]\,.$$

3.8. The sign in this last equality disappears if one changes slightly the point of view: recall that for any DG $\mathcal{R}_*$–modules $\mathcal{M}_*, \mathcal{N}_*$ and any integer k one has a natural isomorphism of DG $\mathcal{R}_*$–modules that moves the shift functor T^k to the second factor,

$$(\mathcal{M}_* \otimes \mathcal{N}_*)[k] \xrightarrow{\cong} \mathcal{M}_* \otimes (\mathcal{N}_*[k])\,, \quad T^k(m \otimes n) \mapsto (-1)^{k|m|} m \otimes T^k n\,,$$

where m is a local homogeneous section of $\mathcal{M}_*$ and n is a local section of $\mathcal{N}_*$.

Composing $\operatorname{At}^k(\mathcal{M}_*)$ with this isomorphism yields then a morphism

$$\mathcal{M}_* \xrightarrow{\operatorname{At}^k(\mathcal{M}_*)} (\mathcal{M}_* \underline{\otimes} \Omega^k_{\mathcal{R}_*/Y})[k] \xrightarrow{\cong} \mathcal{M}_* \underline{\otimes} (\Omega^k_{\mathcal{R}_*/Y}[k]) \xrightarrow{\cong} \mathcal{M}_* \underline{\otimes} \mathbb{S}^k(\Omega^1_{\mathcal{R}_*/Y}[1])$$

that we denote, abusively, again by $\operatorname{At}^k(\mathcal{M}_*)$. In this form, 3.3 and 3.7 translate into the following

Proposition 3.9. *The powers of the Atiyah class define morphisms of exact functors*

$$\operatorname{At}^k(\ ) : \mathrm{id} \longrightarrow (\ ) \underline{\otimes} \mathbb{S}^k(\Omega^1_{\mathcal{R}_*/Y}[1])\,, \quad k \geq 0\,,$$

on $D^-_{coh}(\mathcal{R}_*)$ that commute with the shift functor. □

Atiyah classes of coherent $\mathcal{O}_X$–modules. Now we descend to X. Let $\mathcal{F} \in D^-_{coh}(X)$ be a complex and $(X_*, W_*, \mathcal{R}_*)$ a resolvent of the morphism of complex spaces $X \to Y$. Associating to these data the complex of $\mathcal{O}_{X_*}$–modules $\mathcal{F}_*$ with $\mathcal{F}_\alpha = \mathcal{F}|X_\alpha$, the powers of the Atiyah classes of $\mathcal{F}$ are defined to be the images of $\operatorname{At}^k(\mathcal{F}_*)$ under the isomorphism

$$\operatorname{Ext}^k_{X_*}\left(\mathcal{F}_*, \mathcal{F}_* \underline{\otimes} \Omega^k_{\mathcal{R}_*/Y} \underline{\otimes} \mathcal{O}_{X_*}\right) \cong \operatorname{Ext}^k_X\left(\mathcal{F}, \mathcal{F} \underline{\otimes} \Lambda^k \mathbb{L}_{X/Y}\right)\,,$$

$$\operatorname{At}^k(\mathcal{F}_*) \mapsto \operatorname{At}^k(\mathcal{F})\,.$$

Theorem 3.10. *The Atiyah classes $\operatorname{At}^k(\mathcal{F}) \in \operatorname{Ext}^k_X(\mathcal{F}, \mathcal{F} \otimes \Lambda^k \mathbb{L}_{X/Y})$ are well defined for each $\mathcal{F} \in D^-_{coh}(X)$.*

Proof. To compare the Atiyah classes formed with DG algebra resolutions $\mathcal{R}_*$ and $\mathcal{R}'_*$ of $\mathcal{O}_{X_*}$ over W_*, note first that by 2.32 there is a free $\mathcal{R}_* \otimes_{\mathcal{O}_{W_*}} \mathcal{R}'_*$ algebra that constitutes a resolvent of $\mathcal{O}_{X_*}$. Thus we may suppose that $\mathcal{R}'_*$ is a free $\mathcal{R}_*$-algebra. In this case 3.6 gives the independence from the choice of DG algebra resolution.

The construction is as well independent of the embedding $X_* \subseteq W_*$. With similar arguments as above it suffices to compare two embeddings $X_* \subseteq W_*$ and $X_* \subseteq W'_*$ that are related by a smooth map $p : W'_* \to W_*$, meaning that $\mathcal{O}_{W_\alpha, p(x)} \subseteq \mathcal{O}_{W'_\alpha, x}$ is smooth for every $x \in W'_*$. If $\mathcal{R}_*$ is a free $\mathcal{O}_{W_*}$-algebra forming a resolvent then we can take a free $p^*\mathcal{R}_*$ algebra, say, $\mathcal{R}'_*$ as a resolvent on W_*. Now a projective

$\mathcal{R}_*$-resolution $\mathcal{P}_*$ of $\mathcal{F}_*$ gives a projective $\mathcal{R}'_*$-resolution $\mathcal{P}'_* = \mathcal{P}_* \otimes_{\mathcal{R}_*} \mathcal{R}'_*$, and 3.6 applies again.

Finally, the independence from the choice of locally finite coverings by Stein compact sets is easily seen considering refinements; we leave the simple details to the reader. □

We now translate the earlier results on the naturality of Atiyah classes. The following is an almost immediate consequence of 3.3.

Proposition 3.11. *For every morphism of complexes $\alpha : \mathcal{F} \to \mathcal{G}$ of degree 0 in $D^-_{coh}(X)$ the diagram*

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\alpha"] \arrow[d, "\mathrm{At}^k(\mathcal{F})"'] & \mathcal{G} \arrow[d, "\mathrm{At}^k(\mathcal{G})"] \\ \mathcal{F} \underline{\otimes} \Lambda^k \mathbb{L}_{X/Y} \arrow[r, "\alpha\otimes\mathrm{id}"] & \mathcal{G} \underline{\otimes} \Lambda^k \mathbb{L}_{X/Y} \end{tikzcd}$$

commutes.

Proof. Consider as in 2.33 a resolvent of X over Y and choose projective approximations $\mathcal{P}_* \to \mathcal{F}_*$ and $\mathcal{Q}_* \to \mathcal{G}_*$. There is a morphism $\tilde{\alpha}_* : \mathcal{P}_* \to \mathcal{Q}_*$ lifting the given morphism α and the assertion follows now easily from 3.3. □

For $\mathcal{F} \in D^-_{coh}(X)$ as before, $\mathrm{Ext}^\bullet_X(\mathcal{F},\mathcal{F}) := \bigoplus_i \mathrm{Ext}^i_X(\mathcal{F},\mathcal{F})$ carries a natural algebra structure given by Yoneda product, and $\mathrm{Ext}^\bullet_X(\mathcal{F},\mathcal{F} \underline{\otimes} \Lambda^k \mathbb{L}_{X/Y})$ is a bimodule over $\mathrm{Ext}^\bullet_X(\mathcal{F},\mathcal{F})$.

Proposition 3.12. *The power $\mathrm{At}^k(\mathcal{F})$ of the Atiyah class of $\mathcal{F}$ is a (graded) central element of degree k in the bimodule $\mathrm{Ext}^\bullet_X(\mathcal{F},\mathcal{F} \underline{\otimes} \Lambda^k \mathbb{L}_{X/Y})$, which means that*

$$\xi \cdot \mathrm{At}^k(\mathcal{F}) = (-1)^{ik}\, \mathrm{At}^k(\mathcal{F}) \cdot \xi$$

for every element $\xi \in \mathrm{Ext}^i_X(\mathcal{F},\mathcal{F})$.

Proof. By the preceding result and by 3.7 the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\xi"] \arrow[d, "\mathrm{At}^k(\mathcal{F})"'] & \mathcal{F}[i] \arrow[d, "(-1)^{ik}\,\mathrm{At}^k(\mathcal{F})[i]"] \\ \mathcal{F} \underline{\otimes} \Lambda^k \mathbb{L}_{X/Y} \arrow[r, "\xi\otimes\mathrm{id}"] & \mathcal{F}[i] \underline{\otimes} \Lambda^k \mathbb{L}_{X/Y} \end{tikzcd}$$

commutes. □

Using the isomorphisms $\mathbb{S}^k(\mathbb{L}_{X/Y}[1]) \xrightarrow{\cong} (\Lambda^k \mathbb{L}_{X/Y})[k]$, this compatibility with morphisms can be summarized in analogy to 3.9 as follows.

Corollary 3.13. *The powers of the Atiyah class define morphisms of exact functors*

$$\mathrm{At}^k(\ ) : \mathrm{id} \longrightarrow (\ ) \underline{\otimes} \mathbb{S}^k(\mathbb{L}_{X/Y}[1])\,, \quad k \geq 0\,,$$

on $D^-_{coh}(X)$ that commute with the shift functor. □

The Atiyah classes are as well compatible with mappings in the following sense.

Proposition 3.14. *Let*

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

be a diagram of complex spaces and $\mathcal{F} \in D^-_{coh}(X)$. *Under the natural map*

$$\mathrm{Ext}^k_X(\mathcal{F}, \mathcal{F} \underline{\otimes} \Lambda^k \mathbb{L}_{X/Y}) \longrightarrow \mathrm{Ext}^k_{X'}(Lf^*(\mathcal{F}), Lf^*(\mathcal{F}) \underline{\otimes} \Lambda^k \mathbb{L}_{X'/Y'})$$

the Atiyah class $\mathrm{At}^k(\mathcal{F})$ *is mapped onto the Atiyah class* $\mathrm{At}^k(Lf^*(\mathcal{F}))$.

Before giving the proof we have to choose compatible resolvents for X and X'. This is done as follows. Consider locally finite coverings $\{X_j\}_{j\in J}$, $\{X'_i\}_{i\in I}$ of X resp. X' by Stein compact sets such that there is a map $\sigma : I \to J$ with $f(X'_i) \subseteq X_{\sigma(i)}$ for all $i \in I$. We may assume that $I = J$ and $\sigma = \mathrm{id}_I$ since otherwise we can replace I and J by the disjoint union $K = I \cup J$ and consider the coverings $\{X_k\}_{k\in K}$ resp. $\{X'_k\}_{k\in K}$, where we set $X_i := X_{\sigma(i)}$ for $i \in I$ and $X'_j := \emptyset$ for $j \in J$.

Let $X_* = (X_\alpha)_{\alpha\in A}$ and $X'_* = (X'_\alpha)_{\alpha\in A}$ be the associated simplicial spaces. There is a natural functor $f^* : \mathbf{Coh}(X_*) \to \mathbf{Coh}(X'_*)$ which associates to $\mathcal{M}_* \in \mathbf{Coh}(X_*)$ the module with $f^*(\mathcal{M}_*)_\alpha := (f|X_\alpha)^*(\mathcal{M}_\alpha)$. It is easy to see that f^* transforms projective modules into projective modules (for instance, using 2.13 it is sufficient to check this for modules of type $p^*(\mathcal{P}_\alpha)$).

Choose embeddings $X_i \hookrightarrow L_i$, $X'_i \hookrightarrow V_i$ into Stein compact sets in $\mathbb{C}^{n_i} \times Y$ resp. $\mathbb{C}^{n_i} \times Y'$ and take the diagonal embedding $X'_i \subseteq L'_i := L_i \times_Y V_i$. From the data

$$X_i \subseteq L_i \quad \text{and} \quad X'_i \subseteq L'_i$$

we construct smoothings $X_* \subseteq W_*$, $X'_* \subseteq W'_*$ of $X \to Y$ and $X' \to Y'$, respectively, as explained in 2.2. The projections induce a system of compatible maps $\tilde f : W'_\alpha \to W_\alpha$ restricting to f on X'_α. As above there is a natural functor $\tilde f^* : \mathbf{Coh}\, W_* \to \mathbf{Coh}\, W'_*$ transforming projective modules into projective modules. Therefore, if $\mathcal{R}_* \to \mathcal{O}_{X_*}$ is a resolvent then $\tilde f^*(\mathcal{R}_*)$ is a free DG algebra over W'_*. The projection $\tilde f^*(\mathcal{R}_*) \to \mathcal{O}_{X'_*}$ can be factored through a quasiisomorphism $\mathcal{R}'_* \to \mathcal{O}_{X'_*}$ such that $\mathcal{R}'_*$ is a free DG $\tilde f^*(\mathcal{R}_*)$-algebra, and we take this as a resolvent for X' over Y'.

We remark that one has a natural functor $\tilde f^{-1} : \mathbf{Ab}(W_*) \to \mathbf{Ab}(W'_*)$ on the category of simplicial systems of abelian groups on W_* with $\tilde f^{-1}(\mathcal{A})_\alpha := \tilde f^{-1}(\mathcal{A}_\alpha)$.

After these preparation we can easily deduce 3.14.

Proof of 3.14. Let $\mathcal{F}_* \to \tilde{\mathcal{F}}_*$ be a quasiisomorphism into a W_*-acyclic module, $\mathcal{P}_* \to \tilde{\mathcal{F}}_*$ a projective approximation, and let $\nabla : \mathcal{P}_* \to \mathcal{P}_* \otimes \Omega^1_{\mathcal{R}_*/Y}$ be a connection. Using the product rule, the composed map

$$\tilde f^{-1}\mathcal{P}_* \xrightarrow{\tilde f^{-1}(\nabla)} \tilde f^{-1}\mathcal{P}_* \otimes_{\tilde f^{-1}\mathcal{R}_*} \tilde f^{-1}(\Omega^1_{\mathcal{R}_*/Y}) \hookrightarrow \tilde f^{-1}\mathcal{P}_* \otimes_{\tilde f^{-1}\mathcal{R}_*} \Omega^1_{\mathcal{R}'_*/Y'}$$

can be extended to a connection ∇' on $\mathcal{P}'_* := \tilde f^{-1}\mathcal{P}_* \otimes_{\tilde f^{-1}\mathcal{R}_*} \mathcal{R}'_*$. Hence under the natural map

$$\mathrm{Ext}^k_{\mathcal{R}_*}(\mathcal{P}_*, \mathcal{P}_* \otimes \Omega^k_{\mathcal{R}_*/Y}) \longrightarrow \mathrm{Ext}^k_{\mathcal{R}'_*}(\mathcal{P}'_*, \mathcal{P}'_* \otimes \Omega^k_{\mathcal{R}'_*/Y'})$$

the Atiyah class $\mathrm{At}^k(\mathcal{P}_*)$ maps onto $\mathrm{At}^k(\mathcal{P}'_*)$. Since the module on the left is isomorphic to $\mathrm{Ext}^k_X(\mathcal{F}, \mathcal{F} \underline{\otimes} \Lambda^k \mathbb{L}_{X/Y})$ and the module on the right is isomorphic to $\mathrm{Ext}^k_{X'}(Lf^*\mathcal{F}, Lf^*\mathcal{F} \underline{\otimes} \Lambda^k \mathbb{L}_{X'/Y'})$, the result follows. $\square$

Corollary 3.15. *Let*

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

be a diagram of complex spaces and $\alpha : Lf^*\mathcal{F} \to \mathcal{F}'$ *a morphism of complexes of sheaves, where* $\mathcal{F} \in D^-_{coh}(X)$ *and* $\mathcal{F}' \in D^-_{coh}(X')$*. Then the diagram*

$$\begin{tikzcd} Lf^*\mathcal{F} \arrow[r, "\alpha"] \arrow[d, "Lf^*\operatorname{At}^k(\mathcal{F})"'] & \mathcal{F}' \arrow[d, "\operatorname{At}^k(\mathcal{F}')"] \\ Lf^*\mathcal{F}\underline{\otimes}Lf^*(\Lambda^k\mathbb{L}_{X/Y}) \arrow[r, "\alpha\otimes can"] & \mathcal{F}'\underline{\otimes}\Lambda^k\mathbb{L}_{X'/Y'} \end{tikzcd}$$

commutes.

Proof. This is an immediate consequence of 3.11 and the fact that $Lf^*\operatorname{At}^k(\mathcal{F}) = \operatorname{At}^k(Lf^*\mathcal{F})$. □

In the final result of this section we explain how to compute the Atiyah class using the second fundamental form. Slightly more generally it is convenient to show the following result.

Proposition 3.16. *Let* X *be smooth over* Y *and let*

$$0 \to \mathcal{F}' \xrightarrow{j} \mathcal{F} \xrightarrow{p} \mathcal{F}'' \to 0$$

be an exact sequence of coherent sheaves on X*. Assume that there is a map* $\nabla : \mathcal{F} \to \mathcal{F}'' \otimes \Omega_{X/Y}$ *satisfying the product rule* $\nabla(fa) = \nabla(f)a + p(f) \otimes da$ *for local sections* f *in* $\mathcal{F}$ *and* a *in* $\mathcal{O}_X$*. The linear map*

$$\sigma := \nabla \circ j : \mathcal{F}' \longrightarrow \mathcal{F}'' \otimes \Omega^1_{X/Y}.$$

is then $\mathcal{O}_X$*-linear, and if*

$$\begin{aligned} \delta' &: \operatorname{Hom}(\mathcal{F}', \mathcal{F}'' \otimes \Omega^1_{X/Y}) \to \operatorname{Ext}^1(\mathcal{F}', \mathcal{F}' \otimes \Omega^1_{X/Y}) \\ \delta'' &: \operatorname{Hom}(\mathcal{F}', \mathcal{F}'' \otimes \Omega^1_{X/Y}) \to \operatorname{Ext}^1(\mathcal{F}'', \mathcal{F}'' \otimes \Omega^1_{X/Y}) \end{aligned}$$

denote the boundary operator in the respective long exact Ext-sequences then

$$\delta'(\sigma) = \operatorname{At}(\mathcal{F}') \quad \textit{and} \quad \delta''(\sigma) = -\operatorname{At}(\mathcal{F}'').$$

If $\mathcal{F}$ itself admits a connection, say, $\nabla_1 : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_{X/Y}$, then this result applies to $\nabla := p \otimes 1 \circ \nabla_1$, and the map σ becomes the usual second fundamental form. This shows thus in particular how to compute the Atiyah class using the second fundamental form.

Proof. Let $(X_*, W_*, \mathcal{R}_*)$ be a resolvent of X as in 2.33. There are projective resolutions $\mathcal{P}'_*$, $\mathcal{P}_* \cong \mathcal{P}'_* \oplus \mathcal{P}''_*$, $\mathcal{P}''_*$ of $\mathcal{F}'_*$, $\mathcal{F}_*$, $\mathcal{F}''_*$, respectively that fit into a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{P}'_* \arrow[r, "inj"] \arrow[d, "\pi'"] & \mathcal{P}_* \cong \mathcal{P}'_* \oplus \mathcal{P}''_* \arrow[r, "\tilde{p}"] \arrow[d, "\pi"] & \mathcal{P}''_* \arrow[r] \arrow[d, "\pi''"] & 0 \\ 0 \arrow[r] & \mathcal{F}'_* \arrow[r, "j"] & \mathcal{F}_* \arrow[r, "p"] & \mathcal{F}''_* \arrow[r] & 0 \end{tikzcd}$$

Consider connections ∇', ∇'' on $\mathcal{P}'_*$, $\mathcal{P}''_*$, respectively, and equip $\mathcal{P}_*$ with the connection $\nabla' \oplus \nabla''$. Using the isomorphisms

$$\begin{aligned}\operatorname{Hom}(\mathcal{F}', \mathcal{F}'' \otimes \Omega^1_{X/Y}) &\cong \operatorname{Hom}(\mathcal{P}'_*, \mathcal{F}''_* \otimes \Omega^1_{\mathcal{R}_*/Y})\\ \operatorname{Ext}^1(\mathcal{F}'', \mathcal{F}'' \otimes \Omega^1_{X/Y}) &\cong H^1(\operatorname{Hom}(\mathcal{P}''_*, \mathcal{F}''_* \otimes \Omega^1_{\mathcal{R}_*/Y})),\end{aligned}$$

$\delta''(\sigma)$ can be computed as follows. The linear map $\sigma \circ \pi'$ can be extended to a linear map $\tilde\sigma : \mathcal{P}_* \to \mathcal{F}'' \otimes \Omega^1_{\mathcal{R}_*/Y}$, and $\tilde\sigma \circ \partial$ is zero on $\mathcal{P}'_*$ and so defines a map $\sigma'' : \mathcal{P}''_* \to \mathcal{F}''_* \otimes \Omega^1_{\mathcal{R}_*/Y}$ that represents $-\delta''(\sigma)$. Taking as extension the map $\tilde\sigma := \nabla \circ \pi - (\pi'' \otimes 1) \circ \nabla'' \circ \tilde p$ we get $\tilde\sigma \circ \partial = -(\pi'' \otimes 1) \circ \nabla'' \partial \circ \tilde p$, and so $(\pi'' \otimes 1) \circ \nabla'' \partial$ represents $\delta''(\sigma)$. By construction it also represents $-\operatorname{At}(\mathcal{F}'')$, proving the second part of the result. The equality $\delta'(\sigma) = \operatorname{At}(\mathcal{F}')$ follows with a similar argument and is left to the reader. □

Remarks 3.17. 1. If X is smooth over Y and if $\mathcal{F}$ is a coherent $\mathcal{O}_X$–module then the Atiyah class of $\operatorname{At}(\mathcal{F})$ is the cohomology class that is represented by the extension

$$(*) \qquad 0 \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_{X/Y} \xrightarrow{j} \mathcal{P}^1(\mathcal{F}) := p_1^*(\mathcal{F}) \otimes_{\mathcal{O}_{X\times_Y X}} \mathcal{O}_{X\times_Y X}/\mathcal{J}^2 \xrightarrow{p} \mathcal{F} \to 0\,,$$

where $\mathcal{J} \subseteq \mathcal{O}_{X\times_Y X}$ is the ideal of the diagonal and where we consider $\mathcal{P}^1(\mathcal{F})$ as an $\mathcal{O}_X$–module via the second projection p_2. Using 3.16 a simple proof can be given as follows. The map $\nabla' : p_1^*(\mathcal{F}) \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_{X/Y}$ given by $\nabla(f \otimes a) := f \otimes da$ for local sections f of $\mathcal{F}$, a of $\mathcal{O}_X$, is easily seen to factor through a map $\nabla : \mathcal{P}^1(\mathcal{F}) \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_{X/Y}$ that satisfies the product rule required in 3.16. The map $\sigma := \nabla \circ j$ is just the identity on $\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_X$ as $\Omega^1_{X/Y}$ is identified with $\mathcal{J}/\mathcal{J}^2$ via $da \mapsto 1 \otimes a - a \otimes 1$. Hence 3.16 shows that $\operatorname{At}(\mathcal{F})$ is represented by $-\delta''(\sigma)$, where $\delta'' : \operatorname{Hom}(\mathcal{F} \otimes \Omega^1_{X/Y}, \mathcal{F} \otimes \Omega^1_{X/Y}) \to \operatorname{Ext}^1(\mathcal{F}, \mathcal{F} \otimes \Omega^1_{X/Y})$ is the boundary map. In view of [ALG, X.126, Cor.1(b)] this proves the remark.

2. In order to check that the sign of our Atiyah classes is correct, consider the case that $X = \mathbb{P}^n$ and $\mathcal{M} = \mathcal{O}_{\mathbb{P}^n}(1)$. It is well known that the extension class of the Euler sequence

$$0 \longrightarrow \Omega^1_{\mathbb{P}^n}(1) \xrightarrow{j} \mathcal{O}^{n+1}_{\mathbb{P}^n} = \textstyle\bigoplus_{i=0}^n e_i \mathcal{O}_{\mathbb{P}^n} \xrightarrow{p} \mathcal{O}_{\mathbb{P}^n}(1) \longrightarrow 0$$

in $\operatorname{Ext}^1_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}(1), \Omega^1_{\mathbb{P}^n}(1)) \cong H^1(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}) \cong \mathbb{C}$ represents the first Chern class $c_1(\mathcal{O}_{\mathbb{P}^n}(1))$. Explicitly, the maps j, p are given by $j(x_i dx_j - x_j dx_i) = e_j x_i - e_i x_j$ and $p(e_i) = x_i$ respectively, with $x_0, \dots, x_n$ the homogeneous coordinates of $\mathbb{P}^n$. The module $\mathcal{O}^{n+1}_{\mathbb{P}^n}$ admits a unique connection with $\nabla(e_i) = 0$, and the corresponding map σ in 3.16 is easily seen to be $-\operatorname{id}$ on $\Omega^1_{\mathbb{P}^n}(1)$. Hence it follows from 3.16 that $c_1(\mathcal{O}_{\mathbb{P}^n}(1)) = -\operatorname{At}(\mathcal{O}_{\mathbb{P}^n}(1))$.

Summing up we have a naturally defined class

$$\exp(-\operatorname{At}(\mathcal{F})) = \sum_k (-1)^k \operatorname{At}^k(\mathcal{F})/k! \in \prod_k \operatorname{Ext}^k_X(\mathcal{F}, \mathcal{F} \,\underline{\otimes}\, \Lambda^k \mathbb{L}_{X/Y})\,,$$

the *Atiyah-Chern character*, for every complex $\mathcal{F}$ in $D^-_{coh}(X)$.

4. The semiregularity map

A semiregularity map for modules. Recall that a complex $\mathcal{F}$ over a complex space X is called *perfect* if it admits locally a quasiisomorphism to a bounded complex of free coherent $\mathcal{O}_X$–modules. For instance, every $\mathcal{O}_X$–module on a complex manifold when considered as a complex concentrated in degree 0 is perfect. If $\mathcal{G} \in D(X)$ is a complex then for every perfect complex $\mathcal{F}$ there is a natural trace map

$$\operatorname{Tr} : \operatorname{Ext}^k_X(\mathcal{F}, \mathcal{F} \underline{\otimes} \mathcal{G}) \longrightarrow H^k(X, \mathcal{G}), \quad k \geq 0,$$

see [Ill]. These maps are compatible with taking cup products: if $\xi \in \operatorname{Ext}^j_X(\mathcal{G}, \mathcal{G}')$, where $\mathcal{G}'$ is another complex in $D(X)$, then the diagram

$$\begin{tikzcd} \operatorname{Ext}^k_X(\mathcal{F}, \mathcal{F} \underline{\otimes} \mathcal{G}) \arrow[r, "\operatorname{Tr}"] \arrow[d, "\xi"] & H^k(X, \mathcal{G}) \arrow[d, "\xi"] \\ \operatorname{Ext}^{k+j}_X(\mathcal{F}, \mathcal{F} \underline{\otimes} \mathcal{G}') \arrow[r, "\operatorname{Tr}"] & H^{k+j}(X, \mathcal{G}') \end{tikzcd}$$

commutes. For instance, applying the trace map to the Atiyah classes we obtain for every perfect complex $\mathcal{F}$ well defined classes

$$\operatorname{ch}_k(\mathcal{F}) := \operatorname{Tr}((-1)^k \operatorname{At}^k(\mathcal{F}))/k! \in H^k(X, \Lambda^k \mathbb{L}_{X/Y}),$$

that are the components of the *Chern character* $\operatorname{ch}(\mathcal{F}) = \operatorname{Tr}\exp(-\operatorname{At}(\mathcal{F}))$ of $\mathcal{F}$. If X is a manifold and $\mathcal{F}$ is a vector bundle then this gives the usual Chern character of X, see [At, OTT], and in the general algebraic case it is Illusie's [Ill] description.

Definition 4.1. Let $X \to Y$ be a morphism of complex spaces and let $\mathcal{F}$ be a perfect complex of $\mathcal{O}_X$–modules. The map

$$\sigma := \operatorname{Tr}(* \cdot \exp(-\operatorname{At}(\mathcal{F}))) : \operatorname{Ext}^2_X(\mathcal{F}, \mathcal{F}) \longrightarrow \prod_k H^{k+2}(X, \Lambda^k \mathbb{L}_{X/Y})$$

is called the *semiregularity map* for $\mathcal{F}$.

Slightly more generally, for every coherent $\mathcal{O}_X$–module $\mathcal{N}$ and every $r \geq 0$ there are maps

$$\sigma = \sigma_{\mathcal{N}} := \operatorname{Tr}(* \cdot \exp(-\operatorname{At}(\mathcal{F}))) : \operatorname{Ext}^r_X(\mathcal{F}, \mathcal{F} \underline{\otimes} \mathcal{N}) \longrightarrow \prod_k H^{k+r}(X, \mathcal{N} \underline{\otimes} \Lambda^k \mathbb{L}_{X/Y}),$$

to which we will also refer as the semiregularity map.

To formulate the next proposition, set $\mathbb{L} := \mathbb{L}_{X/Y}$ and note that the group $A := \bigoplus_i A^i$ with

$$A^i := \bigoplus_j \operatorname{Ext}^{i+j}_X(\mathcal{F}, \mathcal{F} \underline{\otimes} \Lambda^j \mathbb{L}) \cong \bigoplus_j \operatorname{Ext}^i_X(\mathcal{F}, \mathcal{F} \underline{\otimes} \mathbb{S}^j(\mathbb{L}[1]))$$

carries a natural algebra structure that is associative but in general not graded commutative. Moreover $M := \bigoplus_i M^i$ with $M^i := \bigoplus_j \operatorname{Ext}^i_X(\mathcal{F}, \mathcal{F} \underline{\otimes} \mathcal{N} \underline{\otimes} \mathbb{S}^j(\mathbb{L}[1]))$ is a graded A-bimodule. Every element $\xi \in T^{r-1}_{X/Y}(\mathcal{N}) \cong \operatorname{Ext}^r_X(\mathbb{L}[1], \mathcal{N})$ defines a derivation $\langle \xi, * \rangle : A \to M$ of degree r which is induced by the composition

$$\Lambda^\bullet \mathbb{L} \xrightarrow{\ \Delta\ } \mathbb{L} \underline{\otimes} \Lambda^\bullet \mathbb{L} \xrightarrow{\xi \otimes \mathrm{id}} \mathcal{N} \underline{\otimes} \Lambda^\bullet \mathbb{L},$$

where Δ is the indicated component of the comultiplication on $\Lambda^\bullet \mathbb{L}$. Thus, for elements $\omega_1 \in A^i$, $\omega_2 \in A$ we have

$$\langle \xi, \omega_1 \omega_2 \rangle = \langle \xi, \omega_1 \rangle \omega_2 + (-1)^{ir} \omega_1 \langle \xi, \omega_2 \rangle.$$

In particular, for $\mathcal{F} = \mathcal{O}_X$, this gives a derivation $\langle \xi, * \rangle$ from the cohomology algebra $\bigoplus H^{i+j}(X, \Lambda^j \mathbb{L})$ into $\bigoplus H^{i+j}(X, \mathcal{N} \underline{\otimes} \Lambda^j \mathbb{L})$. Note that using resolvents one can check all this very explicitly.

Proposition 4.2. *Let $X \to Y$ be a morphism of complex spaces. If $\mathcal{F}$ is a perfect complex on X and $\mathcal{N}$ is an $\mathcal{O}_X$–module then the diagram*

$$\begin{tikzcd} T^{r-1}_{X/Y}(\mathcal{N}) \arrow[rr, "{\langle *, -\operatorname{At}(\mathcal{F}) \rangle}"] \arrow[dr, "{\langle *, \operatorname{ch}_{k+1}(\mathcal{F}) \rangle}"'] & & \operatorname{Ext}^r_X(\mathcal{F}, \mathcal{F} \underline{\otimes} \mathcal{N}) \arrow[dl, "\sigma_k"] \\ & H^{k+r}(X, \mathcal{N} \underline{\otimes} \Lambda^k \mathbb{L}_{X/Y}) & \end{tikzcd}$$

commutes, where σ_k denotes the k^{th} component of $\sigma_{\mathcal{N}}$.

Proof. We need to show for $\xi \in T^{r-1}_{X/Y}(\mathcal{N})$ that

$$\operatorname{Tr}\big(\langle \xi, \operatorname{At}(\mathcal{F}) \rangle \cdot \frac{\operatorname{At}^k(\mathcal{F})}{k!}\big) = \big\langle \xi\, ,\, \operatorname{Tr}\big(\frac{\operatorname{At}^{k+1}(\mathcal{F})}{(k+1)!}\big)\big\rangle\ .$$

As the trace map is compatible with taking cup products, the diagram

$$\begin{tikzcd} \operatorname{Ext}^{k+1}_X(\mathcal{F}, \mathcal{F} \underline{\otimes} \Lambda^{k+1}\mathbb{L}) \arrow[r, "\Delta"] \arrow[d, "\operatorname{Tr}"] & \operatorname{Ext}^{k+1}_X(\mathcal{F}, \mathcal{F} \underline{\otimes} \mathbb{L} \underline{\otimes} \Lambda^k \mathbb{L}) \arrow[r, "\xi"] \arrow[d, "\operatorname{Tr}"] & \operatorname{Ext}^{k+r}_X(\mathcal{F}, \mathcal{F} \underline{\otimes} \mathcal{N} \underline{\otimes} \Lambda^k \mathbb{L}) \arrow[d, "\operatorname{Tr}"] \\ H^{k+1}(X, \Lambda^{k+1}\mathbb{L}) \arrow[r, "\Delta"] & H^{k+1}(X, \mathbb{L} \underline{\otimes} \Lambda^k \mathbb{L}) \arrow[r, "\xi"] & H^{k+r}(X, \mathcal{N} \underline{\otimes} \Lambda^k \mathbb{L}) \end{tikzcd}$$

commutes, where as before $\mathbb{L} = \mathbb{L}_{X/Y}$. Therefore

$$\operatorname{Tr}\big\langle \xi, \frac{\operatorname{At}^{k+1}(\mathcal{F})}{(k+1)!}\big\rangle = \big\langle \xi, \operatorname{Tr}\big(\frac{\operatorname{At}^{k+1}(\mathcal{F})}{(k+1)!}\big)\big\rangle\ . \tag{1}$$

As $\langle \xi, * \rangle$ is a derivation of degree r on $\operatorname{Ext}^{\bullet}_X(\mathcal{F}, \mathcal{F} \underline{\otimes} \mathbb{S}^{\bullet}(\mathbb{L}[1]))$ and $\operatorname{At}^k(\mathcal{F}) \in A^0 \cong \operatorname{Ext}^0_X(\mathcal{F}, \mathcal{F} \underline{\otimes} \mathbb{S}^{\bullet}(\mathbb{L}[1]))$, we obtain that

$$\langle \xi, \operatorname{At}^{k+1}(\mathcal{F}) \rangle = \sum \operatorname{At}^i(\mathcal{F}) \langle \xi, \operatorname{At}(\mathcal{F}) \rangle \operatorname{At}^{k-i}(\mathcal{F}).$$

For homogeneous endomorphisms f, g the trace satisfies $\operatorname{Tr}(fg) = (-1)^{|f||g|} \operatorname{Tr}(gf)$, whence taking traces yields

$$\operatorname{Tr}\big\langle \xi, \frac{\operatorname{At}^{k+1}(\mathcal{F})}{(k+1)!}\big\rangle = \operatorname{Tr}\left(\langle \xi, \operatorname{At}(\mathcal{F}) \rangle \cdot \frac{\operatorname{At}^k(\mathcal{F})}{k!}\right).$$

Comparing with (1), the result follows. ∎

In case that X is smooth and Y is a reduced point the theorem above specializes to the following corollary.

Corollary 4.3. *For every complex manifold X and every coherent $\mathcal{O}_X$–module $\mathcal{F}$ there is a commutative diagram*

$$\begin{tikzcd} H^{r-1}(X, \Theta_X) \arrow[rr, "{\langle *, -\operatorname{At}(\mathcal{F}) \rangle}"] \arrow[dr, "{\langle *, \operatorname{ch}_{k+1}(\mathcal{F}) \rangle}"'] & & \operatorname{Ext}^r_X(\mathcal{F}, \mathcal{F}) \arrow[dl, "\sigma_k"] \\ & H^{k+r}(X, \Omega^k_X)\,. & \end{tikzcd}$$

In case of compact algebraic manifolds the map $\langle *, \mathrm{ch}_{k+1}(\mathcal{F})\rangle$ on $H^1(X,\Theta_X)$ has the following geometric interpretation; see [Blo, 4.2], or, for a more general statement, 5.7, 5.8. Given an infinitesimal deformation of X represented by a class $\xi \in H^1(X,\Theta_X)$, the unique horizontal lift of $\mathrm{ch}_{k+1}(\mathcal{F})$ relative to the Gauß-Manin connection stays of Hodge type $(k+1,k+1)$ if and only if $\langle \xi, \mathrm{ch}_{k+1}(\mathcal{F})\rangle = 0$. In the next result we will show that $\langle \xi, -\mathrm{At}(\mathcal{F})\rangle$ gives the obstruction for deforming $\mathcal{F}$ in the direction of ξ. Thus the semiregularity map σ_k relates the obstruction to deform $\mathcal{F}$ along ξ with the obstruction that its Chern class $\mathrm{ch}_{k+1}(\mathcal{F})$ stays of pure Hodge type along ξ.

Proposition 4.4. *Let $f : X \to Y$ be a morphism of complex spaces and let $X \subseteq X'$ be a Y–extension of X by a coherent $\mathcal{O}_X$–module $\mathcal{N}$ so that X' defines a class $[X'] \in T^1_{X/Y}(\mathcal{N})$. If $\mathcal{F}$ is a coherent $\mathcal{O}_X$–module then under the composed map*

$$\mathrm{ob} : T^1_{X/Y}(\mathcal{N}) \xrightarrow{\langle *, -\mathrm{At}(\mathcal{F})\rangle} \mathrm{Ext}^2_X(\mathcal{F}, \mathcal{F} \underline{\otimes} \mathcal{N}) \xrightarrow{can} \mathrm{Ext}^2_X(\mathcal{F}, \mathcal{F} \otimes \mathcal{N})$$

one has $\mathrm{ob}([X']) = 0$ if and only if there is an $\mathcal{O}_{X'}$–module $\mathcal{F}'$ extending $\mathcal{F}$ to X', i.e. there is an exact sequence of $\mathcal{O}_{X'}$–modules

$$0 \to \mathcal{F} \otimes \mathcal{N} \to \mathcal{F}' \to \mathcal{F} \to 0\,.$$

Proof. In the algebraic case this is shown in [Ill, IV.3.1.8]. In the analytic case we can proceed as follows. Let $(X_*, W_*, \mathcal{R}_*)$ be a resolvent of X over Y as in 2.33 and let $\gamma : \mathcal{P}_* \to \mathcal{F}_*$ be a projective resolution of $\mathcal{F}_*$ as an $\mathcal{R}_*$–module. A Y-extension $[X']$ of X gives rise to an extension $[X'_*]$ of X_* by $\mathcal{N}_*$. Since W_* is smooth over Y, the embedding $X_* \hookrightarrow W_*$ can be lifted to a Y-map $X'_* \hookrightarrow W_*$, and the surjection of algebras $\pi : \mathcal{R}_* \to \mathcal{O}_{X_*}$ to a map of $\mathcal{O}_{W_*}$-algebras $\pi' : \mathcal{R}_* \to \mathcal{O}_{X'_*}$. With ∂ the differential on $\mathcal{R}_*$, the map $\xi := -\pi'\partial : \mathcal{R}_* \to \mathcal{N}_*$ is a Y-derivation of degree 1 that represents the class of $[X']$ in

$$T^1_{X/Y}(\mathcal{N}) \cong H^1(\mathrm{Der}_Y(\mathcal{R}_*, \mathcal{N}_*)) \cong \mathrm{Ext}^1_{\mathcal{R}_*}(\Omega^1_{\mathcal{R}_*/Y}, \mathcal{N}_*).$$

If one equips the trivial extension $\mathcal{R}_*[\mathcal{N}_*]$ with the differential $(r,n) \mapsto (\partial(r), \xi(r))$, the map $\pi' + \mathrm{id}_{\mathcal{N}_*} : \mathcal{R}_*[\mathcal{N}_*] \to \mathcal{O}_{X'_*}$ becomes a quasiisomorphism of DG algebras that restricts to the identity on $\mathcal{N}_*$.

Let now $\nabla : \mathcal{P}_* \to \mathcal{P}_* \otimes \Omega^1_{\mathcal{R}_*/Y}$ be a connection. Contracting with ξ and projecting onto $\mathcal{F}_*$ gives a map $\nabla_\xi : \mathcal{P}_* \to \mathcal{F}_* \otimes \mathcal{N}_*$ of degree 1 satisfying the product rule $\nabla_\xi(pr) = \nabla_\xi(p)r + (-1)^{|p|}\gamma(p) \otimes \xi(r)$ for local sections r in $\mathcal{R}_*$ and p in $\mathcal{P}_*$. The class $\mathrm{ob}(\mathcal{F})$ is represented by the map

$$-(\gamma \otimes \xi) \circ [\partial, \nabla] = [\partial, \nabla_\xi] = \nabla_\xi \partial : \mathcal{P}_* \longrightarrow \mathcal{F}_* \otimes \mathcal{N}_*$$

of degree 2; note that by 2.28, 2.25(3) and 2.23

$$(*) \qquad \mathrm{Ext}^2_X(\mathcal{F}, \mathcal{F} \otimes \mathcal{N}) \cong H^2(\mathrm{Hom}_{\mathcal{R}_*}(\mathcal{P}_*, \mathcal{F}_* \otimes \mathcal{N}_*))\,.$$

If the class $\mathrm{ob}(\mathcal{F})$ vanishes then $[\partial, \nabla_\xi] = [\partial, h]$ for some $\mathcal{R}_*$-linear map $h : \mathcal{P}_* \to \mathcal{F}_* \otimes \mathcal{N}_*$ of degree 1. The differential

$$(p, f \otimes n) \mapsto (\partial(p), (\nabla_\xi - h)(p))$$

defines then on $\mathcal{P}_* \oplus \mathcal{F}_* \otimes \mathcal{N}_*$ the structure of a DG module over $\mathcal{R}_*[\mathcal{N}_*]$ and we denote this DG module $\mathcal{P}_*[\mathcal{F}_* \otimes \mathcal{N}_*]$. Using the exact cohomology sequence associated to the exact sequence of DG modules

$$0 \to \mathcal{F}_* \otimes \mathcal{N}_* \to \mathcal{P}_*[\mathcal{F}_* \otimes \mathcal{N}_*] \to \mathcal{P}_* \to 0\,,$$

it follows that the $\mathcal{O}_{X'_*}$ module $\mathcal{H}^0(\mathcal{P}_*[\mathcal{F}_*\otimes\mathcal{N}_*])$ is an extension of $\mathcal{F}_*$ by $\mathcal{F}_*\otimes\mathcal{N}_*$. Gluing yields an $\mathcal{O}_{X'}$ modules $\mathcal{F}'$ which is an extension of $\mathcal{F}$ by $\mathcal{F}\otimes\mathcal{N}$.

Conversely assume that there exists such an extension $\mathcal{F}'$ of $\mathcal{F}$ by $\mathcal{F}\otimes\mathcal{N}$. As $\mathcal{P}_*$ is projective the map $\gamma:\mathcal{P}_*\to\mathcal{F}_*$ can be lifted to a map of $\mathcal{R}_*$–modules $\gamma':\mathcal{P}_*\to\mathcal{F}'_*$. A simple calculation shows that $\delta:=-\gamma'\partial:\mathcal{P}_*\to\mathcal{F}_*\otimes\mathcal{N}_*$ satisfies the product rule $\delta(pr)=\delta(p)r+(-1)^{|p|}\gamma(p)\otimes\xi(r)$ for sections r in $\mathcal{R}_*$ and p in $\mathcal{P}_*$. Hence $h=\nabla_\xi-\delta$ is $\mathcal{R}_*$-linear and satisfies $[\partial,h]=[\partial,\nabla_\xi]$, whence the cohomology class of $[\partial,\nabla_\xi]$ in $H^2(\mathrm{Hom}(\mathcal{P}_*,\mathcal{F}_*\otimes\mathcal{N}_*))$ vanishes. As this class represents $\mathrm{ob}(\mathcal{F})$ under the isomorphism $(*)$, the result follows. □

Later on we will apply the semiregularity map to modules on the total space of a deformation of a complex space. In order to verify that such a module has locally finite projective dimension, the following standard criterion is useful.

Proposition 4.5. *Let $f:X\to S$ be a flat morphism of complex spaces and $\mathcal{F}$ a complex in $D^-_{coh}(X)$. If the restriction $\mathcal{F}\underline{\otimes}_{\mathcal{O}_X}\mathcal{O}_{X_s}$ to every fibre $X_s:=f^{-1}(s)$ is a perfect complex on X_s, then $\mathcal{F}$ is a perfect complex on X.*

This is an immediately consequence of the following simple lemma from commutative algebra.

Lemma 4.6. *Let $A\to B$ be a flat morphism of local noetherian rings and let $F^\bullet$ be a complex of B–modules with finite cohomology that is bounded above. If $F^\bullet\underline{\otimes}_A A/\mathfrak{m}_A$ is a perfect complex of $B/\mathfrak{m}_AB$–modules then $F^\bullet$ is a perfect complex of B–modules.*

Proof. For the convenience of the reader we include the simple argument. We may assume that $F^\bullet$ is a complex of finite free B–modules with $F^i=0$ for $i\gg 0$. By assumption $F^\bullet\otimes A/\mathfrak{m}_A$ is a perfect complex and so for $k\ll 0$ the complex

$$F^\bullet_{(k)}:\qquad \ldots\to F^{k-1}\xrightarrow{\partial}F^k\to F^k/\partial F^{k-1}\to 0$$

has the property that $F^\bullet_{(k)}\otimes A/\mathfrak{m}_A$ is exact with $(F^k/\partial F^{k-1})\otimes A/\mathfrak{m}_A$ a free $B/\mathfrak{m}_AB$–module. Using induction on n and the long exact cohomology sequences associated to the exact sequences of complexes

$$0\longrightarrow F^\bullet_{(k)}\otimes\mathfrak{m}_A^n/\mathfrak{m}_A^{n+1}\longrightarrow F^\bullet_{(k)}\otimes A/\mathfrak{m}_A^{n+1}\longrightarrow F^\bullet_{(k)}\otimes A/\mathfrak{m}_A^n\longrightarrow 0$$

it follows that $F^\bullet_{(k)}\otimes A/\mathfrak{m}_A^n$ is exact and that $(F^k/\partial F^{k-1})\otimes A/\mathfrak{m}_A^n$ is a free $B/\mathfrak{m}_A^nB$–module. Hence $F^\bullet_{(k)}$ is exact and $F^k/\partial F^{k-1}$ is free as a B–module, proving the lemma. □

Remarks 4.7. 1. We note that the construction of the semiregularity map is compatible with morphisms. More precisely, given a diagram of complex spaces as in 3.14 and a perfect complex $\mathcal{F}$ on X, for any coherent $\mathcal{O}_X$–module $\mathcal{N}$ the diagram

$$\begin{tikzcd}
\mathrm{Ext}^r_X(\mathcal{F},\mathcal{F}\underline{\otimes}\mathcal{N}) \arrow[r, "\sigma"] \arrow[d] & \prod_k H^{k+r}(X,\mathcal{N}\underline{\otimes}\Lambda^k\mathbb{L}_{X/Y}) \arrow[d] \\
\mathrm{Ext}^r_{X'}(Lf^*\mathcal{F},Lf^*\mathcal{F}\underline{\otimes}Lf^*\mathcal{N}) \arrow[r, "\sigma"] & \prod_k H^{k+r}(X',Lf^*\mathcal{N}\underline{\otimes}\Lambda^k\mathbb{L}_{X'/Y'})
\end{tikzcd}$$

commutes. This follows from 3.14 and the fact that the trace map is compatible with taking inverse images.

2. If in 4.1 the support of $\mathcal{F}$ is contained in a closed subset, say, Z of X then the semiregularity map admits a factorization

$$\operatorname{Ext}^r_X(\mathcal{F},\mathcal{F}\underline{\otimes}\mathcal{N}) \xrightarrow{\sigma_Z} \prod_k H^{k+r}_Z(X,\mathcal{N}\underline{\otimes}\Lambda^k\mathbb{L}_{X/Y}) \xrightarrow{can} \prod_k H^{k+r}(X,\mathcal{N}\underline{\otimes}\Lambda^k\mathbb{L}_{X/Y}).$$

This follows as the trace map factors by [Ill] through the local cohomology.

That the powers of the Atiyah classes are graded central elements by 3.12 allows the following glimpse at the relevance of the semiregularity map for deformation problems. Let

$$[\ ,\ ] : \operatorname{Ext}^i_X(\mathcal{F},\mathcal{F}) \times \operatorname{Ext}^j_X(\mathcal{F},\mathcal{F}) \longrightarrow \operatorname{Ext}^{i+j}_X(\mathcal{F},\mathcal{F}) \quad , \quad [\xi,\zeta] := \xi\zeta - (-1)^{ij}\zeta\xi\,,$$

denote the graded Lie algebra structure underlying the Yoneda product on the Ext–algebra. Centrality of the Atiyah classes together with the fact that the trace vanishes on commutators implies then the following result.

Corollary 4.8. *The family of semiregularity maps*

$$\sigma : \operatorname{Ext}^r_X(\mathcal{F},\mathcal{F}) \longrightarrow \prod_k H^{k+r}(X,\Lambda^k\mathbb{L}_{X/Y}) \quad , \quad r \geq 0\,,$$

vanishes on

$$[\operatorname{Ext}^\bullet_X(\mathcal{F},\mathcal{F}),\operatorname{Ext}^\bullet_X(\mathcal{F},\mathcal{F})] \subseteq \operatorname{Ext}^\bullet_X(\mathcal{F},\mathcal{F})\,.$$

$\square$

As is well known, and will be recalled in Section 6 below, the vector space $\operatorname{Ext}^1_X(\mathcal{F},\mathcal{F})$ is the tangent space to a semiuniversal deformation of $\mathcal{F}$, if it is finite dimensional. The obstructions to lift such tangent directions to second order lie in $[\operatorname{Ext}^1_X(\mathcal{F},\mathcal{F}),\operatorname{Ext}^1_X(\mathcal{F},\mathcal{F})] \subseteq \operatorname{Ext}^2_X(\mathcal{F},\mathcal{F})$, and these obstructions are thus annihilated by the semiregularity map.

A semiregularity map for subspaces. Let X be a complex space, $Z \subseteq X$ a closed complex subspace and $\mathcal{N}$ a coherent $\mathcal{O}_X$–module. In this part we will show how to define a semiregularity map on $T^2_{Z/X}(\mathcal{O}_Z \underline{\otimes}_{\mathcal{O}_X} \mathcal{N})$ provided that $\mathcal{O}_Z$ has locally finite projective dimension as an $\mathcal{O}_X$–module. In particular, this will give a generalization of Bloch's semiregularity map to arbitrary subspaces of manifolds. The idea is to define first a map from $T^2_{Z/X}(\mathcal{O}_Z \underline{\otimes} \mathcal{N})$ into $\operatorname{Ext}^2_X(\mathcal{O}_Z,\mathcal{O}_Z\underline{\otimes}\mathcal{N})$ and then to compose this with the semiregularity map for $\mathcal{F} = \mathcal{O}_Z$ as defined in the previous part. The key technical lemma is as follows.

Lemma 4.9. *Let $Z \subseteq X$ be a closed embedding of complex spaces.*

1. *For each complex of $\mathcal{O}_Z$–modules $\mathcal{M}$ there are natural maps*

$$\epsilon^{(k)} : T^k_{Z/X}(\mathcal{M}) \longrightarrow \operatorname{Ext}^k_X(\mathcal{O}_Z,\mathcal{M})\,, \quad k \in \mathbb{Z}\,.$$

In case $\mathcal{M} = \mathcal{O}_Z$, the map $\epsilon^{(\bullet)} : T^\bullet_{Z/X}(\mathcal{O}_Z) \to \operatorname{Ext}^\bullet_X(\mathcal{O}_Z,\mathcal{O}_Z)$ is a morphism of graded Lie algebras.

2. *Let $X \to Y$ be a morphism of complex spaces and let $\mathcal{N}$ be a coherent $\mathcal{O}_X$–module. With $T^{k-1}_{X/Y}(\mathcal{O}_Z \underline{\otimes} \mathcal{N}) \to T^k_{Z/X}(\mathcal{O}_Z \underline{\otimes} \mathcal{N})$ the boundary map in the long tangent cohomology sequence associated to the triple $Z \to X \to Y$, the composition*

$$T^{k-1}_{X/Y}(\mathcal{N}) \xrightarrow{can} T^{k-1}_{X/Y}(\mathcal{O}_Z \underline{\otimes} \mathcal{N}) \to T^k_{Z/X}(\mathcal{O}_Z \underline{\otimes} \mathcal{N}) \xrightarrow{\epsilon^{(k)}} \operatorname{Ext}^k_X(\mathcal{O}_Z,\mathcal{O}_Z\underline{\otimes}\mathcal{N})$$

is given by $\xi \mapsto \langle \xi, -(-1)^k \operatorname{At}(\mathcal{O}_Z)\rangle$.

Proof. Let $(X_*, W_*, \mathcal{R}_*)$ be a resolvent for X over Y as in 2.33 and choose a quasiisomorphism $\mathcal{M}_* \to \widetilde{\mathcal{M}}_*$ into a W_*-acyclic complex of $\mathcal{O}_{Z_*}$–modules, where $\mathcal{M}_*$ and $\mathcal{O}_{Z_*}$ denote the simplicial sheaves on W_* associated to $\mathcal{O}_Z$ and $\mathcal{M}$, respectively. We choose an algebra resolution $\mathcal{R}_* \xrightarrow{i} \mathcal{S}_* \to \mathcal{O}_{Z_*}$ of the composition $\mathcal{R}_* \to \mathcal{O}_{X_*} \to \mathcal{O}_{Z_*}$ so that $\mathcal{S}_*$ is a graded free algebra over $\mathcal{R}_*$ and $\mathcal{S}_* \to \mathcal{O}_{Z_*}$ is a quasiisomorphism of DG algebras. In particular, $\mathcal{S}_*$ is a projective approximation of $\mathcal{O}_{Z_*}$ as an $\mathcal{R}_*$–module, and $\mathcal{S}_* \to \mathcal{S}_* \otimes_{\mathcal{R}_*} \mathcal{O}_{X_*}$ is a quasiisomorphism by 2.25. It follows that $\mathcal{S}_* \otimes_{\mathcal{R}_*} \mathcal{O}_{X_*}$ provides a projective resolution of $\mathcal{O}_{Z_*}$ as $\mathcal{O}_{X_*}$–module. Hence

$$(*) \qquad \operatorname{Ext}^i_X(\mathcal{O}_Z, \mathcal{M}) \cong H^i(\operatorname{Hom}_{\mathcal{R}_*}(\mathcal{S}_*, \widetilde{\mathcal{M}}_*)) \qquad \text{and}$$
$$T^i_{Z/X}(\mathcal{M}) \cong H^i(\operatorname{Hom}_{\mathcal{R}_*}(\Omega^1_{\mathcal{S}_*/\mathcal{R}_*}, \widetilde{\mathcal{M}}_*))$$

for all i. Composing the natural inclusions, see 2.36,

$$(**) \qquad \operatorname{Hom}_{\mathcal{R}_*}(\Omega^1_{\mathcal{S}_*/\mathcal{R}_*}, \widetilde{\mathcal{M}}_*) \hookrightarrow \operatorname{Der}_{\mathcal{R}_*}(\mathcal{S}_*, \widetilde{\mathcal{M}}_*) \hookrightarrow \operatorname{Hom}_{\mathcal{R}_*}(\mathcal{S}_*, \widetilde{\mathcal{M}}_*)$$

gives the desired map in (1). If $\mathcal{M} = \mathcal{O}_Z$, then $T^\bullet_{Z/X}(\mathcal{O}_Z) \cong H^\bullet(\operatorname{Der}_{\mathcal{R}_*}(\mathcal{S}_*, \mathcal{S}_*))$ and $\operatorname{Ext}^\bullet_X(\mathcal{O}_Z, \mathcal{M}) \cong H^\bullet(\operatorname{End}_{\mathcal{R}_*}(\mathcal{S}_*))$, and the inclusion of derivations into endomorphisms is a morphism of DG Lie algebras that induces a morphism of graded Lie algebras in cohomology.

To show (2), note first that $\mathcal{O}_{Z_*} \underline{\otimes} \mathcal{N}_*$ is represented by $\mathcal{S}_* \otimes_{\mathcal{R}_*} \mathcal{N}_*$. Consider a derivation $\delta \in \operatorname{Der}_Y(\mathcal{R}_*, \mathcal{N}_*)$ of degree $k-1$ that represents the cohomology class ξ in $T^{k-1}_{X/Y}(\mathcal{N})$. Its image in $T^{k-1}_{X/Y}(\mathcal{O}_Z \underline{\otimes} \mathcal{N})$ is then represented by $1 \otimes \delta \in \operatorname{Der}_Y(\mathcal{R}_*, \mathcal{S}_* \otimes \mathcal{N}_*)$. Under the isomorphism in $(*)$, the image of the latter element in $T^k_{Z/X}(\mathcal{O}_Z \underline{\otimes} \mathcal{N})$ is represented by a $\mathcal{R}_*$-derivation $[\partial, \tilde{\delta}]$, where $\tilde{\delta} : \mathcal{S}_* \to \mathcal{S}_* \otimes \mathcal{N}_*$ is a derivation restricting to $1 \otimes \delta$ on $\mathcal{R}_*$. Let now $\nabla : \mathcal{S}_* \to \mathcal{S}_* \otimes \Omega^1_{\mathcal{R}_*/Y}$ be a connection on $\mathcal{S}_*$, as exists by 3.1. If h is one of the maps $(1 \otimes \delta) \circ \nabla$ or $\tilde{\delta}$ from $\mathcal{S}_*$ to $\mathcal{S}_* \otimes \mathcal{N}_*$ then the product rule

$$h(sr) = h(s)r + (1)^{|s|(k-1)} s\delta(r)$$

is satisfied for local sections s in $\mathcal{S}_*$ and r in $\mathcal{R}_*$. It follows that the difference $\tilde{\delta} - (1 \otimes \delta) \circ \nabla$ is $\mathcal{R}_*$-linear and so $[\partial, \tilde{\delta}]$ and $[\partial, (1 \otimes \delta) \circ \nabla]$ represent the same cohomology class in $H^k(\operatorname{Hom}_{\mathcal{R}_*}(\mathcal{S}_*, \mathcal{S}_* \otimes \mathcal{N}_*))$. As

$$\langle [\delta], \operatorname{At}(\mathcal{O}_Z) \rangle = (1 \otimes \delta) \circ [\partial, \nabla] = (-1)^{k-1} [\partial, (1 \otimes \delta) \circ \nabla]\,,$$

by definition, and as δ is of degree $k-1$, we have

$$\xi \mapsto [\partial, \tilde{\delta}] = (-1)^{k-1} \langle [\delta], \operatorname{At}(\mathcal{O}_Z) \rangle = \langle \xi, -(-1)^k \operatorname{At}(\mathcal{O}_Z) \rangle\,,$$

as required. □

Definition 4.10. Let $X \to Y$ be a morphism of complex spaces and let $Z \subseteq X$ be a closed complex subspace of X such that $\mathcal{O}_Z$ has locally finite projective dimension over $\mathcal{O}_X$. The composition of the canonical map $T^2_{Z/X}(\mathcal{O}_Z) \to \operatorname{Ext}^2_X(\mathcal{O}_Z, \mathcal{O}_Z)$ of 4.9 (1) with the semiregularity map σ defined in 4.1 yields a map

$$\tau : T^2_{Z/X}(\mathcal{O}_Z) \longrightarrow \prod_{k \geq 0} H^{k+2}(X, \Lambda^k \mathbb{L}_{X/Y}),$$

which we call the *semiregularity map* for Z.

Again we have such a semiregularity map more generally for any coherent $\mathcal{O}_X$–module $\mathcal{N}$ and any $r \geq 0$,

$$\tau_{\mathcal{N}} := \sigma_{\mathcal{N}} \circ \epsilon^r : T^r_{Z/X}(\mathcal{O}_Z \underline{\otimes} \mathcal{N}) \longrightarrow \prod_{k\geq 0} H^{k+r}(X, \mathcal{N} \underline{\otimes}_{\mathcal{O}_X} \Lambda^k \mathbb{L}_{X/Y}).$$

Note that this gives in particular a semiregularity map for every closed subspace Z of a complex manifold X. Combining 4.2 and 4.9 we obtain the following result.

Proposition 4.11. *With the notations and assumptions as in* 4.10*, the diagram*

$$\begin{tikzcd} T^{r-1}_{X/Y}(\mathcal{N}) \arrow[rr, "can"] \arrow[dr, "{\langle *, \mathrm{ch}_k(\mathcal{O}_Z)\rangle}"'] & & T^r_{Z/X}(\mathcal{O}_Z \underline{\otimes} \mathcal{N}) \arrow[dl, "(-1)^r \tau_{k-1}"] \\ & H^{k+r-1}(X, \mathcal{N} \underline{\otimes} \Lambda^{k-1}\mathbb{L}_{X/Y}), & \end{tikzcd}$$

commutes, where τ_{k-1} is the $(k-1)^{st}$ component of the semiregularity map $\tau_{\mathcal{N}}$.

For instance, for Y a reduced point, X smooth, and Z a closed subscheme of codimension k in X, we can rephrase 4.11 as follows.

Corollary 4.12. *For a complex manifold X and a subspace Z of codimension k the diagram*

$$\begin{tikzcd} H^{r-1}(X, \Theta_X) \arrow[rr, "can"] \arrow[dr, "{\langle *, [Z]\rangle}"'] & & T^r_{Z/X}(Z) \arrow[dl, "(-1)^r \tau_{k-1}"] \\ & H^{k+r-1}(X, \Omega^{k-1}_X), & \end{tikzcd}$$

commutes, where $[Z]$ denotes the fundamental class of Z in $H^k(X, \Omega^k_X)$.

Proof. For a manifold, $T^{r-1}_{X/Y}(\mathcal{O}_X) \cong H^{r-1}(X, \Theta_X)$, and according to Grothendieck, [Gro, 4(16)], see also [Mur, 2.18], the fundamental class of Z is given by

$$[Z] = \frac{(-1)^{k-1}}{(k-1)!} c_k(\mathcal{O}_Z) = \mathrm{ch}_k(\mathcal{O}_Z),$$

whence the result follows from 4.11. □

Let $X \to Y$ be a morphism of complex spaces and let $Z \subseteq X$ be a closed complex subspace of X. In the final result of this section we consider for a coherent $\mathcal{O}_X$–module $\mathcal{N}$ the boundary map δ in the long exact tangent cohomology sequence

$$\cdots \longrightarrow T^1_{Z/Y}(\mathcal{O}_Z \otimes \mathcal{N}) \longrightarrow T^1_{X/Y}(\mathcal{O}_Z \otimes \mathcal{N}) \xrightarrow{\delta} T^2_{Z/X}(\mathcal{O}_Z \otimes \mathcal{N}) \to \cdots$$

for the triple $Z \to X \to Y$. The exactness of this sequence gives immediately the following interpretation of the composed map

$$\gamma : T^1_{X/Y}(\mathcal{N}) \xrightarrow{can} T^1_{X/Y}(\mathcal{O}_Z \otimes \mathcal{N}) \xrightarrow{\delta} T^2_{Z/X}(\mathcal{O}_Z \otimes \mathcal{N})$$

in terms of extensions.

Lemma 4.13. *Let X' be an extension of X by $\mathcal{N}$. The image of the class $[X'] \in T^1_{X/Y}(\mathcal{N})$ under γ vanishes, $\gamma([X']) = 0$, if and only if there is an extension Z' of*

Z by $\mathcal{O}_Z \otimes \mathcal{N}$ that fits into a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{N} \arrow[r] \arrow[d, "can"'] & \mathcal{O}_{X'} \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_Z \otimes \mathcal{N} \arrow[r] & \mathcal{O}_{Z'} \arrow[r] & \mathcal{O}_Z \arrow[r] & 0\,.
\end{tikzcd}$$

□

Remarks 4.14. 1. The construction of the canonical map $\epsilon^{(\bullet)}$ in 4.9 (1) is compatible with morphisms of complex spaces. More precisely, assume given a diagram of complex spaces as in 3.14 and a closed subspace Z of X. With $Z' := f^{-1}(Z) \subseteq X'$, the diagram

$$\begin{tikzcd}
T^r_{Z/X}(\mathcal{M}) \arrow[r, "\epsilon^{(r)}"] \arrow[d] & \mathrm{Ext}^r_X(\mathcal{O}_Z, \mathcal{M}) \arrow[d] \\
T^r_{Z'/X'}(\mathcal{O}_{Z'} \underline{\otimes}_{\mathcal{O}_Z} \mathcal{M}) \arrow[r, "\epsilon^{(r)}"] & \mathrm{Ext}^r_{X'}(\mathcal{O}_{Z'}, \mathcal{O}_{Z'} \underline{\otimes}_{\mathcal{O}_Z} \mathcal{M})
\end{tikzcd}$$

commutes for every coherent $\mathcal{O}_Z$–module $\mathcal{M}$.

2. In analogy with 4.7, the construction of the semiregularity map τ is also compatible with morphisms of complex spaces. This follows from 4.7, using the preceding remark. We leave the straightforward formulation and its proof to the reader.

3. It follows from 4.7(2) that the semiregularity map τ factors through local cohomology,

$$T^r_{Z/X}(\mathcal{O}_Z \underline{\otimes} \mathcal{N}) \xrightarrow{\tau_Z} \prod_k H^{k+r}_Z(X, \mathcal{N} \underline{\otimes} \Lambda^k \mathbb{L}_{X/Y}) \xrightarrow{can} \prod_k H^{k+r}(X, \mathcal{N} \underline{\otimes} \Lambda^k \mathbb{L}_{X/Y})\,.$$

Finally, we wish to point out that 4.8 carries over as well to the family of semiregularity maps for subspaces, the comparison map $\epsilon^{(\bullet)} : T^\bullet_{Z/X}(\mathcal{O}_Z) \to \mathrm{Ext}^\bullet_X(\mathcal{O}_Z, \mathcal{O}_Z)$ being a morphism of graded Lie algebras.

5. Applications to the variational Hodge conjecture

Let X be a compact complex algebraic manifold so that its cohomology admits a Hodge decomposition

$$H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^q(X, \Omega^p_X)\,.$$

Recall that a cohomology class

$$\alpha \in H^{p,p}(X, \mathbb{Q}) := H^p(X, \Omega^p_X) \cap H^p(X, \mathbb{Q})$$

is called *algebraic* if an appropriate multiple $k\alpha, k \in \mathbb{N}$, is represented by an algebraic cycle Z of codimension p in X, in the sense that $k\alpha = [Z]$, the cohomology class of the cycle. The famous and so far unsolved Hodge conjecture asks wether every class in $H^{p,p}(X, \mathbb{Q})$ is algebraic.

In [Gro], Grothendieck proposed the following weaker version that is called the *variational Hodge conjecture*. Let $\pi : X \to S$ be a deformation of a compact algebraic manifold $X_0 = \pi^{-1}(0)$ over a smooth germ $(S, 0)$. The local system $R^{2p} f_*(\mathbb{C}) \otimes \mathcal{O}_S$ carries then the natural *Gauß-Manin connection*. Assume that α

is a horizontal section of $R^p f_*(\Omega^p_{X/S})$ in the sense that α can be lifted locally to a horizontal section in $R^{2p} f_*(\Omega^{\geq p}_{X/S}) \subseteq R^{2p} f_*(\mathbb{C}) \otimes \mathcal{O}_S$, see also 5.4.

The variational Hodge conjecture asks now: If the restriction of α to the special fibre, $\alpha(0) \in H^p(X_0, \Omega^p_{X_0})$, is algebraic, is then $\alpha(s) \in H^p(X_s, \Omega^p_{X_s})$ algebraic for all $s \in S$ near 0, where $X_s := \pi^{-1}(s)$?

In this section we will give an affirmative answer to this problem if the class $\alpha(0)$ is the p^{th} component $\mathrm{ch}_p(\mathcal{E}_0)$ of the Chern character of some coherent sheaf $\mathcal{E}_0$ for which the p^{th} component of the semiregularity map,

$$\mathrm{Ext}^2_{X_0}(\mathcal{E}_0, \mathcal{E}_0) \longrightarrow H^{p+1}(X_0, \Omega^{p-1}_{X_0}),$$

is injective. We will then call $\mathcal{E}_0$ in brief a *p-semiregular sheaf*. Slightly more generally, with n the dimension of X_0 it is convenient to introduce for any subset $I \subseteq \{0, \dots n\}$ the following notion: $\mathcal{E}_0$ is called *I-semiregular* if the part of the semiregularity map

$$\sigma_I : \mathrm{Ext}^2_{X_0}(\mathcal{E}_0, \mathcal{E}_0) \longrightarrow \prod_{p \in I} H^{p+1}(X_0, \Omega^{p-1}_{X_0})$$

is injective. The main result of this section is the following theorem.

Theorem 5.1. *Let $\pi : X \to S$ be a deformation of a compact complex algebraic manifold X_0 over a smooth germ $S = (S, 0)$ and set $X_s := \pi^{-1}(s)$ for $s \in S$. Assume that $(\alpha_p)_{p \in I}$ is a horizontal section in $\prod_{p \in I} R^p \pi_*(\Omega^p_{X/S})$. If there is an I-semiregular sheaf $\mathcal{E}_0$ on X_0 with*

$$\alpha_p(0) = \mathrm{ch}_p(\mathcal{E}_0) \in H^p(X_0, \Omega^p_{X_0}), \quad p \in I,$$

then $\alpha_p(s) \in H^p(X_s, \Omega^p_{X_s})$ is algebraic for all $s \in S$ near 0 and each $p \in I$.

In analogy with the notion of a I-semiregular sheaf on X_0, a complex subspace $Z_0 \subseteq X_0$ will be called *I-semiregular* if the part of the semiregularity map

$$\tau_I : T^2_{Z_0/X_0}(\mathcal{O}_{Z_0}) \longrightarrow \prod_{p \in I} H^{p+1}(X_0, \Omega^{p-1}_{X_0})$$

is injective. We will also derive the following variant of 5.1 that generalizes a result of S. Bloch [Blo].

Theorem 5.2. *Let $\pi : X \to S$ and $(\alpha_p)_{p \in I}$ be as in* 5.1. *If there is an I-semiregular subspace $Z_0 \subseteq X_0$ with $\alpha_p(0) = \mathrm{ch}_p(\mathcal{O}_{Z_0})$ for $p \in I$ then $\alpha_p(s)$ is algebraic for all $s \in S$ near 0 and each $p \in I$.*

For the proof of these results we need a few preparations.

Lemma 5.3. *Let $X \subseteq X'$ be an extension of a complex space X by a coherent $\mathcal{O}_X$–module $\mathcal{M}$ and assume that the morphism $\xi : \mathbb{L}_{X/X'} \to \mathcal{M}$ of degree 1 in the derived category $D(X)$ represents the class*

$$[X'] \in T^1_{X/X'}(\mathcal{M}) \cong \mathrm{Ext}^1_X(\mathbb{L}_{X/X'}, \mathcal{M}).$$

If $\mathbb{L}_{X/X'} \xrightarrow{+1} \mathbb{L}_{X'} \underline{\otimes} \mathcal{O}_X$ denotes the canonical map of degree 1, then the diagram

$$\begin{tikzcd} \mathbb{L}_{X/X'} \arrow[r, "+1"] \arrow[d, "+1"', "\xi"] & \mathbb{L}_{X'} \underline{\otimes} \mathcal{O}_X \arrow[d, "can"] \\ \mathcal{M} \arrow[r, "d"] & \Omega^1_{X'} \otimes \mathcal{O}_X \end{tikzcd}$$

commutes in $D(X)$.

Proof. Let $(X'_*, W'_*, \mathcal{R}'_*)$ be a resolvent of X' and choose a free graded DG $\mathcal{R}'_*$–algebra $\mathcal{R}_*$ that provides a DG algebra resolution $p : \mathcal{R}_* \to \mathcal{O}_{X_*}$ of the composition $\mathcal{R}'_* \xrightarrow{p'} \mathcal{O}_{X'_*} \to \mathcal{O}_{X_*}$. By 2.25, the induced map $\mathcal{R}_* \otimes_{\mathcal{R}'_*} \mathcal{O}_{X'_*} \to \mathcal{O}_{X_*}$ is a quasiisomorphism and so $(X_*, W_* := X'_*, \mathcal{R}_* \otimes_{\mathcal{R}'_*} \mathcal{O}_{X'_*})$ constitutes a resolvent for X over X'. Thus

$$(*) \qquad T^1_{X/X'}(\mathcal{M}) \cong H^1(\mathrm{Der}_{\mathcal{O}_{X'_*}}(\mathcal{R}_* \otimes_{\mathcal{R}'_*} \mathcal{O}_{X'_*}, \mathcal{M}_*)) \cong H^1(\mathrm{Der}_{\mathcal{R}'_*}(\mathcal{R}_*, \mathcal{M}_*))\,,$$

and moreover

$$\mathbb{L}_{X/X'} \cong C^\bullet(\Omega^1_{\mathcal{R}_* \otimes_{\mathcal{R}'_*} \mathcal{O}_{X'_*}/\mathcal{O}_{X'_*}} \otimes \mathcal{O}_{X_*}) \cong C^\bullet(\Omega^1_{\mathcal{R}_*/\mathcal{R}'_*} \otimes \mathcal{O}_{X_*})\,,$$

where $C^\bullet$ is the Čech functor as in 2.27. By construction, $\mathcal{R}_*$ is a free $\mathcal{R}'_*$-algebra and so admits an augmentation, say, $\beta : \mathcal{R}_* \to \mathcal{R}'_*$. Clearly, the $\mathcal{R}'$-linear map β will not be a morphism of DG algebras in general, but the commutative diagram

$$\begin{tikzcd}
\mathcal{R}_* \arrow[r, dotted, "\beta"] & \mathcal{R}'_* \arrow[r, "p'"] \arrow[d] & \mathcal{O}_{X'} \arrow[d] \\
& \mathcal{R}_* \arrow[r, "p"] & \mathcal{O}_X\,.
\end{tikzcd}$$

shows that the $\mathcal{R}'_*$-derivation $-p'\beta\partial : \mathcal{R}_* \to \mathcal{O}_{X'}$ factors through a derivation $\xi : \mathcal{R}_* \to \mathcal{M}_*$ that in turn represents $[X'] \in T^1_{X/X'}(\mathcal{M})$ under the isomorphism $(*)$, see [Fle1, (3.13)]. The map $\mathbb{L}_{X/X'} \xrightarrow{+1} \mathbb{L}_{X'} \otimes \mathcal{O}_X$ is induced by the map of complexes

$$[\partial, d\beta] : \Omega^1_{\mathcal{R}_*/\mathcal{R}'_*} \longrightarrow \Omega^1_{\mathcal{R}'_*} \otimes \mathcal{R}_*\,,$$

and composition with the canonical map $\Omega^1_{\mathcal{R}'_*} \otimes \mathcal{R}_* \to \Omega^1_{X'_*} \otimes \mathcal{O}_{X_*}$ yields the map $-dp' \circ d\beta \circ \partial$. As this map coincides with the $\mathcal{R}'_*$-derivation $d \circ \xi : \mathcal{R}_* \to \mathcal{M}_* \xrightarrow{d} \Omega^1_{X'_*} \otimes \mathcal{O}_{X_*}$ under the identification

$$\mathrm{Hom}_{\mathcal{R}_*}(\Omega^1_{\mathcal{R}_*/\mathcal{R}'_*}, \Omega^1_{X'_*} \otimes \mathcal{O}_{X_*}) \cong \mathrm{Der}_{\mathcal{R}'_*}(\mathcal{R}_*, \Omega^1_{X'_*} \otimes \mathcal{O}_{X_*})\,,$$

the result follows. □

5.4. Let $\pi : X \to S$ be a deformation of a Kähler manifold X_0 over an artinian germ $S = (S, 0)$. The canonical exact sequence

$$0 \longrightarrow \mathcal{O}_X \otimes_{\mathcal{O}_S} \Omega^1_S \longrightarrow \Omega^1_X \longrightarrow \Omega^1_{X/S} \longrightarrow 0$$

induces a morphism $\Omega^1_{X/S} \xrightarrow{+1} \mathcal{O}_X \otimes \Omega^1_S$ of degree 1 in the derived category $D(X)$ and as well, by taking exterior powers, morphisms

$$\nabla_{X/S} : \Omega^p_{X/S} \xrightarrow{+1} \Omega^{p-1}_{X/S} \otimes_{\mathcal{O}_S} \Omega^1_S\,, \quad p \geq 1,$$

that we call the *Gauß-Manin connections*. These maps induce the classical Gauß-Manin connections

$$\nabla_{X/S} : H^q(X, \Omega^p_{X/S}) \longrightarrow H^{q+1}(X, \Omega^{p-1}_{X/S}) \otimes_{\mathcal{O}_S} \Omega^1_S.$$

Recall that the spectral sequence

$$E_1^{pq} = H^q(X, \Omega^p_{X/S}) \Rightarrow H^{p+q}(X, \Omega^\bullet_{X/S}) \cong H^{p+q}(X_0, \mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_S$$

degenerates and that, by Griffiths' transversality theorem, the canonical connection ∇ on $H^{p+q}(X,\Omega^\bullet_{X/S})$ satisfies

$$\nabla(H^q(X,\Omega^{\geq p}_{X/S}))\subseteq H^{q+1}(X,\Omega^{\geq p-1}_{X/S})\otimes_{\mathcal{O}_S}\Omega^1_S$$

and induces just the map $\nabla_{X/S}$ above (see [Gri2]). We will call a class α in $H^q(X,\Omega^p_{X/S})$ in brief *horizontal* if it can be lifted to a horizonal section in

$$H^{p+q}(X,\Omega^{\geq p}_{X/S})\subseteq H^{p+q}(X_0,\mathbb{C})\otimes_{\mathbb{C}}\mathcal{O}_S.$$

In particular, the following well known result holds.

Lemma 5.5. *A horizontal class $\alpha\in H^q(X,\Omega^p_{X/S})$ satisfies $\nabla_{X/S}(\alpha)=0$.* □

5.6. Let $\pi:X\to S$ be as in 5.4 and assume that $S\hookrightarrow S'$ is an extension of the artinian germ S by a coherent $\mathcal{O}_S$–module $\mathcal{N}$ such that there is a smooth morphism $\pi':X'\longrightarrow S'$ restricting to π over S. *We will suppose henceforth that the map*

$$(*)\qquad d:\mathcal{N}\longrightarrow\Omega^1_{S'}\otimes_{\mathcal{O}_{S'}}\mathcal{O}_S$$

is injective. Let

$$\nabla':=1_{\mathcal{O}_S}\otimes_{\mathcal{O}_{S'}}\nabla_{X'/S'}:\Omega^q_{X/S}\longrightarrow\Omega^{q-1}_{X/S}\otimes_{\mathcal{O}_{S'}}\Omega^1_{S'}$$

be the map induced by the Gauß- Manin connection for $X'\to S'$ and denote by the same symbol the induced map in cohomology,

$$\nabla':H^p(X,\Omega^q_{X/S})\longrightarrow H^{p+1}(X,\Omega^{q-1}_{X/S})\otimes_{\mathcal{O}_{S'}}\Omega^1_{S'}\,.$$

Lemma 5.7. ([Blo]) *With notation and assumptions as in* 5.6*, a horizontal class $\alpha\in H^q(X,\Omega^p_{X/S})$ can be lifted to a horizontal section in $H^q(X',\Omega^p_{X'/S'})$ if and only if $\nabla'(\alpha)=0$ in $H^{q+1}(X,\Omega^{p-1}_{X/S})\otimes_{\mathcal{O}_{S'}}\Omega^1_{S'}$.* □

Now return to the notation and assumption in 5.4. The extension X' of X by $\mathcal{N}_X:=\mathcal{O}_X\otimes_{\mathcal{O}_S}\mathcal{N}$ gives a class $\xi\in T^1_X(\mathcal{N}_X)$ or, equivalently, a morphism $\xi:\mathbb{L}_X\longrightarrow\mathcal{N}_X$ of degree 1 in the derived category $D(X)$. Taking exterior powers yields a map, denoted by the same symbol,

$$\xi:\Lambda^p\mathbb{L}_X\longrightarrow\Lambda^{p-1}\mathbb{L}_X\underline{\otimes}\mathcal{N}_X\xrightarrow{can}\Omega^{p-1}_{X/S}\otimes\mathcal{N}_X\,.$$

The next result describes how this map relates to the map ∇' introduced in 5.6.

Lemma 5.8. *The diagram*

$$\begin{tikzcd}
\Lambda^p\mathbb{L}_X \arrow[r, "can"] \arrow[d, "\xi", "+1"'] & \Omega^p_{X/S} \arrow[d, "\nabla'", "+1"'] \\
\Omega^{p-1}_{X/S}\otimes\mathcal{N} \arrow[r, "1\otimes d"] & \Omega^{p-1}_{X/S}\otimes_{\mathcal{O}_{S'}}\Omega^1_{S'}
\end{tikzcd}$$

in $D(X)$ is commutative.

Proof. For suitable representative of the cotangent complexes involved there is a commutative diagram of exact sequences of complexes of $\mathcal{O}_X$–modules

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{L}_{X/X'}[-1] \arrow[r] \arrow[d, "\xi"'] & \mathbb{L}_{X'}\underline{\otimes}\mathcal{O}_X \arrow[r] \arrow[d] & \mathbb{L}_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{N}_X \arrow[r, "d"] & \Omega^1_{X'}\otimes\mathcal{O}_X \arrow[r] & \Omega^1_X \arrow[r] & 0\,,
\end{tikzcd}$$

by 5.3 and our assumption that d is injective on $\mathcal{N}$. In $D(X)$ this gives rise to a commutative diagram

$$(\dagger)\qquad \begin{tikzcd} \mathbb{L}_X \arrow[r] \arrow[d] & \mathbb{L}_{X/X'} \arrow[d, "\xi"] \\ \Omega^1_X \arrow[r] & \mathcal{N}_X[1]\,. \end{tikzcd}$$

As well, the commutative diagram of exact sequences of $\mathcal{O}_X$–modules

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{N}_X \arrow[r, "1\otimes d"] \arrow[d, "1\otimes d"'] & \Omega^1_{X'}\otimes\mathcal{O}_X \arrow[r] \arrow[d, equal] & \Omega^1_X \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{O}_X\otimes_{\mathcal{O}_{S'}}\Omega^1_{S'} \arrow[r] & \Omega^1_{X'}\otimes\mathcal{O}_X \arrow[r] & \Omega^1_{X/S} \arrow[r] & 0 \end{tikzcd}$$

yields in $D(X)$ a commutative diagram

$$\begin{tikzcd} \Omega^1_X \arrow[r] \arrow[d] & \mathcal{N}_X[1] \arrow[d, "1\otimes d"] \\ \Omega^1_{X/S}\cong\Omega^1_{X'/S'}\otimes\mathcal{O}_X \arrow[r] & \mathcal{O}_X\otimes_{\mathcal{O}_{S'}}\Omega^1_{S'}[1]\,. \end{tikzcd}$$

Combining this square with the one in ($\dagger$) gives the result for $p = 1$. The general case follows from this by taking appropriate exterior powers. $\square$

The key observation in proving 5.1 is the following proposition. We keep the notation and assumptions as introduced in 5.4 and 5.6.

Proposition 5.9. *Assume $\mathcal{E}$ is a coherent S–flat sheaf on X such that $\mathcal{E}_0 := \mathcal{E}|X_0$ is I-semiregular, where $X_0 = \pi^{-1}(0)$. The following conditions are then equivalent.*

1. *The sheaf $\mathcal{E}$ can be extended to a deformation $\mathcal{E}'$ on X' over S'.*
2. *The partial Chern character $\mathrm{ch}_I(\mathcal{E}) := (\mathrm{ch}_p(\mathcal{E}))_{p\in I} \in \prod_{p\in I} H^p(X,\Omega^p_{X/S})$ can be lifted to a horizontal section in $\prod_{p\in I} H^p(X',\Omega^p_{X'/S'})$.*

Proof. Let $\xi \in T^1_X(\mathcal{N}_X)$ be the class corresponding to the extension $X\hookrightarrow X'$, given by a morphism $\xi : \mathbb{L}_{X/X'} \to \mathcal{N}_X$ in the derived category. The result 5.8 induces a commutative diagram

$$\begin{tikzcd} \prod_{p\in I} H^p(X,\Lambda^p\mathbb{L}_X) \arrow[r] \arrow[d, "\xi"] & \prod_{p\in I} H^p(X,\Omega^p_{X/S}) \cong \prod_{p\in I} H^p(X,\Omega^p_{X'/S'})\otimes_{\mathcal{O}_{S'}}\mathcal{O}_S \arrow[d, "\nabla'"] \\ \prod_{p\in I} H^{p+1}(X,\Omega^{p-1}_{X/S})\otimes_{\mathcal{O}_S}\mathcal{N} \arrow[r, hook, "1\otimes d"] & \prod_{p\in I} H^{p+1}(X,\Omega^{p-1}_{X/S})\otimes_{\mathcal{O}_{S'}}\Omega^1_{S'}\ , \end{tikzcd}$$

where ∇' is the Gauß-Manin connection.

The relative partial Chern character $\mathrm{ch}_I(\mathcal{E}) \in \prod_{p\in I} H^p(X,\Omega^p_{X/S})$ is the image of the absolute partial Chern character $\mathrm{ch}'_I(\mathcal{E}) := (\mathrm{Tr}((-\mathrm{At}(\mathcal{E}))^p/p!))_{p\in I}$ in $\prod_{p\in I} H^p(X,\Lambda^p\mathbb{L}_X)$. Using 5.7 and the injectivity of $1\otimes d$ in the diagram above, it follows that (2) is equivalent to

3. Contracting against ξ sends the absolute partial Chern character to zero,

$$\langle \xi, \mathrm{ch}'_I(\mathcal{E})\rangle = 0\,.$$

Now, 4.2 yields a commutative diagram

$$\begin{tikzcd} T^1_X(\mathcal{N}_X) \arrow[rr, "\mathrm{ob}:=\langle *, -\mathrm{At}(\mathcal{N}_X)\rangle"] \arrow[dr, "\langle *, \mathrm{ch}'_I(\mathcal{E})\rangle"'] & & \mathrm{Ext}^2_X(\mathcal{E}, \mathcal{E}\otimes_{\mathcal{O}_S}\mathcal{N}) \arrow[dl, "\sigma_{I,\mathcal{N}}"] \\ & \prod_{p\in I} H^{p+1}(X, \Omega^{p-1}_{X/S})\otimes_{\mathcal{O}_S}\mathcal{N} & \end{tikzcd}$$

and $\sigma_{I,\mathcal{N}}$ is injective by 5.10 below. It follows that $\mathrm{ob}(\xi) = 0$ if and only if (3) holds. On the other hand, by 4.4, the condition $\mathrm{ob}(\xi) = 0$ is equivalent to (1), so the result follows. $\square$

Lemma 5.10. *With notation as above, if $\mathcal{E}_0$ is I-semiregular then the map $\sigma_{I,\mathcal{N}}$ is injective.*

The *proof* follows by a simple induction on the length of $\mathcal{N}$. The initial step for $\mathcal{N} \cong \mathbb{C}$ is equivalent to the assumption as $\sigma_{I,\mathbb{C}}$ is the corresponding partial semiregularity map for $\mathcal{E}_0$ in view of the isomorphisms

$$\mathrm{Ext}^2_X(\mathcal{E}, \mathcal{E}\otimes_{\mathcal{O}_S}\mathbb{C}) \cong \mathrm{Ext}^2_{X_0}(\mathcal{E}_0, \mathcal{E}_0) \quad , \quad \text{as } \mathcal{E} \text{ is } S\text{–flat,}$$
$$H^{p+1}(X, \Omega^{p-1}_{X/S})\otimes_{\mathcal{O}_S}\mathbb{C} \cong H^{p+1}(X_0, \Omega^{p-1}_{X_0}) \quad , \quad \text{by base change.}$$

The induction step is left to the reader. $\square$

Proof of Theorem 5.1. Let S_n be the n^{th} infinitesimal neighbourhood of 0 in S so that $S_n \hookrightarrow S_{n+1}$ is an extension of S_n by $\mathcal{N}_n = \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}$, where $\mathfrak{m} \subseteq \mathcal{O}_{S,0}$ is the maximal ideal. The map

$$d : \mathcal{N}_n \longrightarrow \Omega^1_{S_{n+1}} \otimes \mathcal{O}_{S_n}$$

is injective and applying 5.9 repeatedly we see that $\mathcal{E}_0$ can be lifted to a deformation $\mathcal{E}_n$ on X_n for all n. Let $\mathcal{E}$ be a versal deformation of $\mathcal{E}_0$ which is a coherent module on $X \times_S T$, where $(T, 0)$ is a complex space germ over $(S, 0)$. Using versality there are $(S, 0)$–morphisms $\varphi_n : (S_n, 0) \to (T, 0)$ with $(1 \times \varphi_n)^*(\mathcal{E}) \cong \mathcal{E}_n$ and $\varphi_{n+1}|S_n = \varphi_n$. Hence $(T, 0) \to (S, 0)$ admits a formal section, namely $\bar\varphi := \varinjlim \varphi_n : (\hat S, 0) \to (T, 0)$. By Artin's approximation theorem, we can find a convergent section $\varphi : (S, 0) \to (T, 0)$. Now $\mathcal{F} := (1 \times \varphi)^*(\mathcal{E})$ is a coherent S-flat module on X that induces $\mathcal{E}_0$ on the special fibre. The uniqueness of the horizontal lifting gives that $\alpha_p = \mathrm{ch}_p(\mathcal{F})$ as sections in $R^p f_*(\Omega^p_{X/S})$ for each $p \in I$. Hence $\alpha_p(s) = \mathrm{ch}_p(\mathcal{F}|X_s)$ is algebraic for all $s \in S$ near 0 and each $p \in I$. $\square$

The *proof* of 5.2 is similar. The sole difference is that in order to derive the analogue of 5.9, one has to use 4.11 and 4.13 instead of 4.2 and 4.4. We leave the technical details to the reader. $\square$

6. Deformation theory

Before formulating the main results we review some basic notation and facts about deformation theories. In contrast to [Sch] we will not use the language of deformation functors but instead employ deformation groupoids as in [Rim, Fle2, BFl]. Most deformations will take place over $\mathbf{An}_\Sigma$, $\mathbf{Art}_\Sigma$, or $\hat{\mathbf{An}}_\Sigma$, the categories of germs of complex spaces, Artinian complex spaces, or formal complex spaces respectively, over a fixed germ $\Sigma = (\Sigma, 0)$.

6.1. Let $p\colon \mathbf{E} \to \mathbf{B}$ be a functor between categories. We will denote objects of $\mathbf{B}$ by capital letters whereas the objects of $\mathbf{E}$ will be written in lower case. To indicate that a is an object of $\mathbf{E}$ over $S \in \mathbf{B}$ we write simply $a \mapsto S$, although this is *not* a morphism!

Recall that a morphism $a' \to a$ over $f : S' \to S$ is *cartesian* if every morphism $b \to a$ over f factors uniquely into $b \xrightarrow{\tilde{g}} a' \to a$ with $p(\tilde{g}) = \mathrm{id}_{S'}$. If $a' \to a$ is a cartesian morphism over $f\colon S' \to S$, one sets, slightly abusively, $a \times_S S' := a'$.

Following [Rim], a *fibration in categories* is a functor $p\colon \mathbf{E} \to \mathbf{B}$ with the following properties:

(FC1) For every morphism $f\colon S' \to S$ in $\mathbf{B}$ and every object a over S there is a morphism $a' \to a$ over f that is cartesian.

(FC2) Compositions of cartesian morphisms are cartesian.

The category $\mathbf{B}$ is often called the *basis* of the fibration. For $S \in \mathbf{B}$ the *fibre* $\mathbf{E}(S)$ is the subcategory of $\mathbf{E}$ whose objects are those $a \in \mathbf{E}$ with $p(a) = S$, and whose morphisms φ are the ones over id_S, that is $p(\varphi) = \mathrm{id}_S$.

Recall that a groupoid is a category in which all morphisms are isomorphisms. A fibration in categories $p\colon \mathbf{E} \to \mathbf{B}$ is called a *fibration in groupoids* if each fiber $\mathbf{E}(S)$ is a groupoid.

These notions provide a natural framework for deformation theory. As an example let us consider deformations of complex spaces.

Example 6.2. Let $\mathbf{E}$ be the category whose objects are the germs of flat holomorphic maps $f\colon (X, X_0) \to (S, 0)$, where $X_0 = f^{-1}(0)$. The morphisms into a second object given by $g\colon (Y, Y_0) \to (T, 0)$ are all cartesian squares

$$\begin{tikzcd} (X, X_0) \arrow[r, "f"] \arrow[d] & (S, 0) \arrow[d] \\ (Y, Y_0) \arrow[r, "g"] & (T, 0)\,. \end{tikzcd}$$

The functor $p\colon \mathbf{E} \to \mathbf{An}_\Sigma$ assigns to f its basis $(S, 0)$.

In a similar way one treats (flat) deformations of singularities, coherent sheaves, or embedded deformations.

6.3. Let $\mathbf{B}$ be one of the categories $\mathbf{An}_\Sigma$, $\mathbf{Art}_\Sigma$ or $\hat{\mathbf{An}}_\Sigma$. Abusing notation again, we write $a \hookrightarrow a'$ when the underlying morphism in $\mathbf{B}$ is an embedding. A fibration in groupoids $p\colon \mathbf{E} \to \mathbf{B}$ is a *deformation theory* if the following homogeneity property is satisfied.

(H) For every diagram in $\mathbf{E}$,

$$\begin{tikzcd} a \arrow[r, hook] \arrow[d] & a' \\ b & \end{tikzcd} \qquad \text{over} \qquad \begin{tikzcd} S \arrow[r, hook, "i"] \arrow[d, "\alpha"'] & S' \\ T & \end{tikzcd}$$

with $i : S \hookrightarrow S'$ an extension by a coherent $\mathcal{O}_{S,0}$–module $\mathcal{M}$ and $\alpha : S \to T$ a finite map of germs, the fibred sum $b' = a' \coprod_a b$ exists in $\mathbf{E}$.

We remark that b' lies necessarily over $S' \coprod_S T$, which in turn exists as an analytic germ by [Schu].

The condition of homogeneity can be weakened to so called *semihomogeneity*, see [Rim]. We remark that the main applications of this section remain true under this weaker condition in view of the results of [Fle2]. Note, however, that in all reasonable geometric situations condition (H) above is satisfied.

If $a_0 \in \mathbf{E}(0)$ is a specific object over the reduced point, then a *deformation* of a_0 over a germ $S = (S, 0)$ is an object $a \in \mathbf{E}(S)$ together with a morphism $a_0 \to a$ that lies necessarily over $0 \hookrightarrow (S, 0)$.

Let $\mathbf{B}$ again be one of the categories $\mathbf{Art}_\Sigma$, $\mathbf{An}_\Sigma$ or $\hat{\mathbf{An}}_\Sigma$ and let $p\colon \mathbf{E} \to \mathbf{B}$ be a fibred groupoid, $S = (S, 0)$ a germ of a complex space and $a \in \mathbf{E}(S)$ an object over S. For a coherent $\mathcal{O}_S$–module $\mathcal{M}$ an *extension of a by $\mathcal{M}$* is a morphism $a \hookrightarrow b$ such that the underlying morphism $S \hookrightarrow T := p(b)$ is an extension of S by $\mathcal{M}$. Two extensions $a \hookrightarrow b$ and $a \hookrightarrow b'$ are said to be isomorphic if there is a morphism $b \to b'$ that is compatible with $a \hookrightarrow b$, $a \hookrightarrow b'$ and induces the identity on $\mathcal{M}$. We denote by $\mathrm{Ex}(a, \mathcal{M})$ the set of such isomorphism classes.

In contrast, consider extensions $a \hookrightarrow b$ with $p(b) = S[\mathcal{M}]$, the trivial extension. Two such extensions $\beta = (a \hookrightarrow b)$ and $\beta' = (a \hookrightarrow b')$ will be called $S[\mathcal{M}]$-isomorphic if there is a morphism of extensions $b \to b'$ over $\mathrm{id}_{S[\mathcal{M}]}$. The corresponding set of isomorphism classes will be denoted $\mathrm{Ex}(a/S, \mathcal{M})$. This vector space is commonly called the space of *infinitesimal deformations* (of first order).

We recall the following facts, see e.g. [Fle1, Fle2].

Proposition 6.4. 1. *The vector spaces* $\mathrm{Ex}(a, \mathcal{M})$, $\mathrm{Ex}(a/S, \mathcal{M})$ *define (covariant) functors with respect to* $\mathcal{M} \in \mathbf{Coh}(S)$. *They are compatible with direct products: for coherent* $\mathcal{O}_S$*–modules* $\mathcal{M}, \mathcal{N}$ *and for* F *one of these functors, one has naturally* $F(\mathcal{M} \times \mathcal{N}) \xrightarrow{\sim} F(\mathcal{M}) \times F(\mathcal{N})$.

2. *The functors in* (1) *are* $\mathcal{O}_S$*-linear, i.e. they carry natural* $\mathcal{O}_S$*–module structures. Moreover they are half-exact.*

3. *There is a functorial exact sequence*

$$T^0_{S/\Sigma}(\mathcal{M}) \xrightarrow{\delta_{KS}} \mathrm{Ex}(a/S, \mathcal{M}) \to \mathrm{Ex}(a, \mathcal{M}) \to T^1_{S/\Sigma}(\mathcal{M})\,.$$

□

Note that a functor $G\colon\ \mathbf{Coh}(S) \to \mathbf{Sets}$ compatible with finite direct products and satisfying $G(0) \neq \emptyset$ always carries a natural $\mathcal{O}_S$–module structure, whence the first part of (2) is a consequence of (1). Moreover, if $G'\colon\ \mathbf{Coh}(S) \to \mathbf{Sets}$ is a second such functor, and if $G \to G'$ is a morphism of functors, then the maps $G(\mathcal{M}) \to G'(\mathcal{M})$ are necessarily $\mathcal{O}_S$-linear.

We will refer to the sequence in (3) as the Kodaira-Spencer sequence. Moreover, δ_{KS} is the so called *Kodaira-Spencer map.*

6.5. Let $p\colon \mathbf{E} \to \mathbf{Art}_\Sigma$ be a deformation theory and consider its completion $\hat{p}\colon \hat{\mathbf{E}} \to \hat{\mathbf{An}}_\Sigma$, as described (dually) in [Rim]. Recall that a deformation $a \in \hat{\mathbf{E}}(S)$ is called *formally versal* if it satisfies the following lifting property: for every morphism $b \hookrightarrow b'$ in $\hat{\mathbf{E}}$ lying over a closed embedding $T \hookrightarrow T'$, and for every map $f\colon b \to a$, there is a morphism $f'\colon b' \to a$ lifting f. Moreover, a is said to be *formally semiuniversal* if the induced map of tangent spaces $T_{T',0} \to T_{S,0}$ is independent of the lifting.

By the theorem of Schlessinger (see [Rim]), if $\mathrm{Ex}(a_0, \mathbb{C})$ is a vector space of finite dimension then a formally versal deformation of a_0 exists. Moreover, we have the following criterion for formal versality.

Proposition 6.6. ([Fle2]) *Let a be a deformation of a_0 over the base $S \in \widehat{\mathbf{An}}_\Sigma$. The following statements are equivalent.*

1. *The deformation a is formally versal.*
2. $\mathrm{Ex}(a, \mathcal{M}) = 0$ *for every finite $\mathcal{O}_S$–module $\mathcal{M}$.*
3. $\mathrm{Ex}(a, \mathcal{O}_S/\mathfrak{m}_S) = 0$.

In case S/Σ is smooth, $T^1_{S/\Sigma}(\mathcal{M})$ vanishes for every $\mathcal{M}$ and so, in view of the Kodaira-Spencer sequence, a is formally versal if and only if the Kodaira-Spencer map $\delta_{KS}\colon T^0_{S/\Sigma}(\mathcal{O}_S/\mathfrak{m}_S) \to \mathrm{Ex}(a/S, \mathcal{O}_S/\mathfrak{m}_S)$ is surjective.

Next, we give a simple proof of a result by Z. Ran [Ran1]. We state more generally a relative version over an arbitrary base Σ. To formulate the last part of it, recall that an artinian germ $T \in \mathbf{An}_\Sigma$ is *curvilinear* if $\mathcal{O}_T \cong \mathbb{C}[\![X]\!]/(X^n)$ as local $\mathbb{C}$–algebras.

Theorem 6.7. *If $a_0 \in \mathbf{E}(0)$ admits a formally semiuniversal deformation $a \in \hat{\mathbf{E}}(S)$ over some formal germ S, then the following conditions are equivalent.*

1. *The germ $(S, 0)$ is smooth over a closed subspace of the completion $(\hat{\Sigma}, 0)$.*
2. *The functor $\mathcal{M} \longmapsto \mathrm{Ex}(a/S, \mathcal{M})$ is right exact on $\mathbf{Coh}(S)$.*
3. *For every $b \in \mathbf{E}(T)$ over an artinian germ $T \in \mathbf{An}_\Sigma$, the map of infinitesimal deformations*
$$\mathrm{Ex}(b/T, \mathcal{O}_T) \longrightarrow \mathrm{Ex}(b/T, \mathcal{O}_T/\mathfrak{m}_T)$$
is surjective.

Moreover, if Σ is a reduced point, then these are equivalent to the following condition.

4. *The map in (3) is surjective for every $b \in \mathbf{E}(T)$ over an artinian curvilinear germ $T \in \mathbf{An}_\Sigma$, .*

Proof. For (1)⇒(2) observe that the map $T^0_{S/\Sigma}(\mathcal{M}) \to \mathrm{Ex}(a/S, \mathcal{M})$ is surjective due to the versality of a. In turn, by 6.8 below, the functor $\mathcal{M} \mapsto T^0_{S/\Sigma}(\mathcal{M}) \cong \mathrm{Der}_\Sigma(\mathcal{O}_S, \mathcal{M})$ is right exact as S is smooth over a subspace of $\hat{\Sigma}$. Thus, $\mathrm{Ex}(a/S, \mathcal{M})$ is right exact in $\mathcal{M}$ as well. To show (2)⇒(3), let b, T be as in (3). By versality, there is a morphism $b \to a$ that lies over some Σ–morphism $T \to S$. If now $b \hookrightarrow b'$ is an extension of b over $T \hookrightarrow T[\mathcal{M}]$, then the homogeneity condition yields an extension $a \hookrightarrow a' := a \coprod_b b'$ over $S \hookrightarrow S[\mathcal{M}]$ that satisfies $b' = a' \times_S T$. Thus, there is a natural isomorphism
$$\mathrm{Ex}(a/S, \mathcal{M}) \cong \mathrm{Ex}(b/T, \mathcal{M})$$
for every artinian $\mathcal{O}_T$–module $\mathcal{M}$, whence (2) implies (3). In order to show (3)⇒(1) consider the n^{th} infinitesimal neighbourhood S_n of 0 in S and set $a_n := a \times_S S_n$. Repeating the argument just given, we have $\mathrm{Ex}(a/S, \mathcal{M}) \cong \mathrm{Ex}(a_n/S_n, \mathcal{M})$ for any $\mathcal{M} \in \mathbf{Coh}(S_n)$. Hence there is a commutative diagram
$$\begin{array}{ccccc}
T^0_{S/\Sigma}(\mathcal{O}_{S_n}) & \xrightarrow{(\delta_{KS})_n} & \mathrm{Ex}(a/S, \mathcal{O}_{S_n}) & \cong & \mathrm{Ex}(a_n/S_n, \mathcal{O}_{S_n}) \\
{\scriptstyle \alpha_n}\big\downarrow & & {\scriptstyle \beta_n}\big\downarrow & & {\scriptstyle \beta_n}\big\downarrow \\
T^0_{S/\Sigma}(\mathcal{O}_{S_0}) & \xrightarrow{(\delta_{KS})_0} & \mathrm{Ex}(a/S, \mathcal{O}_{S_0}) & \cong & \mathrm{Ex}(a_n/S_n, \mathcal{O}_{S_0})\,.
\end{array}$$
Since a is formally semiuniversal, the map $(\delta_{KS})_0$ is bijective and $(\delta_{KS})_n$ is surjective. By assumption β_n is surjective and so α_n is surjective too. Now the result follows from the Jacobian criterion 6.8 below.

It is obvious that (3) implies (4). Finally, (4)⇒(1) follows with the same reasoning as above from the smoothness criterion given in 6.8 (5). □

In the proof above we have referred to the following smoothness criteria that are essentially reformulations of the Jacobian criterion.

Lemma 6.8. *For a morphism $A \to B$ of complete analytic $\mathbb{C}$-algebras the following conditions are equivalent.*

1. *There is an ideal $\mathfrak{a} \subseteq A$ and an isomorphism of A–algebras*
$$B \cong (A/\mathfrak{a})[\![T_1, \dots T_k]\!] \quad \textit{for some } k \geq 0\,;$$
2. *The B–module $\Omega^1_{B/A}$ is free;*
3. *The functor $M \mapsto \mathrm{Der}_A(B, M)$ is right exact on finite B–modules;*
4. *With $B_n := B/\mathfrak{m}_B^{n+1}$, the natural map $\mathrm{Der}_A(B, B_n) \to \mathrm{Der}_A(B, \mathbb{C})$ is surjective for all n.*

If $A = \mathbb{C}$, then these conditions are also equivalent to:

5. *The natural map $\mathrm{Der}_{\mathbb{C}}(B, C) \to \mathrm{Der}_{\mathbb{C}}(B, \mathbb{C})$ is surjective for any artinian curvilinear B-algebra $C \cong \mathbb{C}[\![X]\!]/(X^n)$.*

Proof. The implications (1)⇒(2)⇒(3)⇒(4)⇒(5) are obvious. To show (4)⇒(1), note first that
$$\mathrm{Der}_A(B, B) \cong \varprojlim \mathrm{Der}_A(B, B_n)$$
as B is complete. Hence (4) implies that $\mathrm{Der}_A(B, B) \to \mathrm{Der}_A(B, \mathbb{C})$ is surjective. This means that there are derivations $\delta_1, \dots, \delta_r : B \to B$ and elements $x_1, \dots, x_r \in \mathfrak{m}_B$ such that $\det(\delta_i(x_j)) \notin \mathfrak{m}_B$, where $r := \dim_{\mathbb{C}} \mathrm{Der}_A(B, \mathbb{C})$. By the criterion of Lipman and Zariski, see [Mat, 30.1], there is then an isomorphism $B \cong C[\![X_1, \dots, X_r]\!]$, where C is an A–subalgebra of B. By construction, $\mathrm{Der}_A(C, \mathbb{C}) = 0$, and so C must be a quotient of A, which gives (1).

Finally, assume that $A = \mathbb{C}$ and that (5) is satisfied. Writing $B = R/I$ with $R := \mathbb{C}[\![X_1, \dots, X_r]\!]$ and $I \subseteq \mathfrak{m}_R^2$, we need to show that $I = 0$. If not, choose a power series $f \neq 0$ of minimal order in I. After a linear change of coordinates we may assume that $f = X_1^n + g$ with $g \in (X_2, \dots, X_r)\mathfrak{m}_R^{n-1}$. Now consider the curvilinear B-algebra $C = B/(X_2, \dots, X_r) \cong \mathbb{C}[\![X_1]\!]/(X_1^n)$. The derivation $\partial/\partial X_1$ on R induces a derivation $B \to \mathbb{C}$ that by assumption can be lifted to a derivation $B \to C$. Composing with $R \to B$ yields a derivation, say, $\delta : R \to C$ with $\delta(I) = 0$. Using the product rule and the fact that $\delta(X_i) \equiv 0 \bmod \mathfrak{m}_C$ for $i \geq 2$, it follows that $\delta(g) = 0$. On the other hand, $\delta(X_1) \equiv 1 \bmod \mathfrak{m}_C$, whence
$$\delta(f) = nX_1^{n-1}\delta(X_1) = nX_1^{n-1} \cdot 1_C \neq 0 \quad \text{in } C\,,$$
and this is a contradiction. □

Remark 6.9. In case $A \neq \mathbb{C}$, the condition (5) above is no longer equivalent to the other ones as the following example shows. Consider a $\mathbb{C}$-algebra $A \cong R/I$, with $R \cong \mathbb{C}[\![X_1, \dots, X_k]\!]$, and assume that there is an element $a \in R$ that is integral over I but not in I. If $\bar{a}$ denotes the residue class of a in A, then the reader may verify that the A-algebra $B := A[\![T]\!]/(\bar{a}T)$ satisfies (5) although it is not smooth over a quotient of A.

Definition 6.10. An *obstruction theory for* $a \in \hat{\mathbf{E}}(S)$ over a formal germ $S \in \hat{\mathbf{An}}_\Sigma$ consists in a functor

$$\mathrm{Ob}(a,-) : \mathbf{Coh}_{art}(S) \longrightarrow \mathbf{Coh}_{art}(S),$$

satisfying the following condition:

Ob1: There is a morphism of functors

$$\mathrm{ob} : T^1_{S/\Sigma}(\mathcal{M}) \longrightarrow \mathrm{Ob}(a,\mathcal{M}) \quad , \quad \mathcal{M} \in \mathbf{Coh}_{art}(S)\,,$$

so that the sequence

$$\mathrm{Ex}(a,\mathcal{M}) \longrightarrow T^1_{S/\Sigma}(\mathcal{M}) \xrightarrow{\mathrm{ob}} \mathrm{Ob}(a,\mathcal{M})$$

is exact for each $\mathcal{M}$.

In other words, if a Σ-extension $S \hookrightarrow S'$ of S by $\mathcal{M} \in \mathbf{Coh}_{art}(S)$ is given, then $\mathrm{ob}([S']) = 0$ in $\mathrm{Ob}(a,\mathcal{M})$ if and only if we can find an extension $a \hookrightarrow a'$ of a by $\mathcal{M}$ over $S \hookrightarrow S'$. As an immediate consequence of the Kodaira-Spencer sequence we obtain the following standard estimate.

Corollary 6.11. *Let* $a \in \hat{\mathbf{E}}(S)$ *be a formally semiuniversal deformation of* a_0 *and assume that there is an obstruction theory for* a*. With*

$$k := \dim_{\mathbb{C}} \mathrm{Ex}(a_0,\mathbb{C}) \quad \textit{and} \quad t := \dim_{\mathbb{C}} \mathrm{Ob}(a,\mathbb{C})\,,$$

the basis S *of* a *can be realized as a subspace of the formal germ* $(\Sigma \times \mathbb{C}^k, 0)\hat{}$ *cut out by at most* t *equations. In particular,*

$$\dim S \geq \dim \Sigma + k - t\,.$$

Proof. As was already observed in the proof of 6.7, we have $\mathrm{Ex}(a_0,\mathbb{C}) \cong \mathrm{Ex}(a/S,\mathbb{C})$, and, as a is formally semiuniversal, the map $T^0_{S/\Sigma}(\mathbb{C}) \to \mathrm{Ex}(a/S,\mathbb{C})$ is bijective. Hence we can write $\mathcal{O}_{S,0} \cong \Lambda[\![X_1,\dots,X_k]\!]/I$ with $\Lambda := \mathcal{O}_{\hat{\Sigma},0}$. By definition, $T^1_{S/\Sigma}(\mathbb{C}) \cong \mathrm{Hom}(I/\mathfrak{m}I,\mathbb{C})$, and so the dimension of $T^1_{S/\Sigma}(\mathbb{C})$ is just the minimal number of generators of I. In the extended Kodaira-Spencer sequence

$$\dots \longrightarrow \mathrm{Ex}(a,\mathbb{C}) \longrightarrow T^1_{S/\Sigma}(\mathbb{C}) \longrightarrow \mathrm{Ob}(a,\mathbb{C})$$

the module $\mathrm{Ex}(a,\mathbb{C})$ vanishes by the versality of a. Hence $T^1_{S/\Sigma}(\mathbb{C})$ injects into $\mathrm{Ob}(a,\mathbb{C})$ and the minimal number of generators of I is bounded by t. □

Let S be a formal germ over $(\Sigma,0)$ and let $\Lambda := \mathcal{O}_{\Sigma,0}$, $A := \mathcal{O}_{S,0}$ denote the associated local rings. It is well known that for a coherent $\mathcal{O}_S$–module $\mathcal{M}$ the group $T^1_{S/\Sigma}(\mathcal{M})$ is canonically isomorphic to $T^1_{A/\Lambda}(M)$, the group of Λ-algebra extensions of A by $M := \mathcal{M}_0$. This group always contains $\mathrm{Ext}^1_A(\Omega^1_{A/\Lambda}, M)$ that can be identified as the subgroup of those extensions whose associated Jacobi map is injective, see 9.2. In case $M = \mathbb{C}$, it contains in turn a further distinguished subspace, $\mathrm{Ex}^c_{A/\Lambda}(\mathbb{C})$, the space of curvilinear extensions, see 9.1 and 9.2. With these notations, the following improvement of 6.7, due to Kawamata, can also easily be deduced.

Corollary 6.12. *Let* $a \in \hat{\mathbf{E}}(S)$ *be a formally semiuniversal deformation of* a_0 *that admits an obstruction theory* $\mathrm{Ob}(a,-)$*. If* $V := \mathrm{ob}(\mathrm{Ex}^c_{A/\Lambda}(\mathbb{C})) \subseteq \mathrm{Ob}(a,\mathbb{C})$ *is the subspace of curvilinear obstructions, then*

$$\dim S \geq \dim \Sigma + \dim_{\mathbb{C}} \mathrm{Ex}(a_0,\mathbb{C}) - \dim_{\mathbb{C}} V\,.$$

Proof. Note that $V \cong \mathrm{Ex}^c_{A/\Lambda}(\mathbb{C})$ as $T^1_{S/\Sigma}(\mathbb{C})$ injects into $\mathrm{Ob}(a,\mathbb{C})$. By 9.2, the vector space $\mathrm{Ex}^c_{A/\Lambda}(\mathbb{C})$ is dual to $I/(J+\mathfrak{m}I)$, where I is as in the proof of 6.11 and J is the integral closure of $\mathfrak{m}I$ in I. Now the result follows from 9.3. $\square$

In the proofs of the preceding two corollaries we only used the module $\mathrm{Ob}(a,\mathbb{C})$ so that it would have been sufficient to have just this module of obstructions at our disposal. However, the next result requires the full strength of the notion of obstruction theory.

Proposition 6.13. *Let $p : \mathbf{E} \to \mathbf{An}_\Sigma$ be a deformation theory and $a \in \hat{\mathbf{E}}(S)$ an object over a germ $S \in \hat{\mathbf{An}}_\Sigma$ that admits an obstruction theory $\mathcal{M} \mapsto \mathrm{Ob}(a,\mathcal{M})$. If there exists a transformation $\psi : \mathrm{Ob}(a,\mathcal{M}) \to G(\mathcal{M})$ into a left exact functor G on $\mathbf{Coh}_{art}(S)$ then the following hold.*

1. *The restriction of $\psi \circ \mathrm{ob}$ to $\mathrm{Ext}^1_S(\Omega^1_{S/\Sigma},\mathcal{M}) \hookrightarrow T^1_{S/\Sigma}(\mathcal{M})$ is the zero map. In other words, if $S \hookrightarrow S'$ is an extension of S by $\mathcal{M} \in \mathbf{Coh}_{art}(S)$ such that the Jacobi map $\mathcal{M} \longrightarrow \Omega^1_{S'/\Sigma} \otimes \mathcal{O}_S$ is injective, then $\psi\,(\mathrm{ob}[S']) = 0$.*
2. *If a is formally versal then $\dim S \geq \dim_{\mathbb{C}} \mathrm{Ex}(a_0,\mathbb{C}) - \dim_{\mathbb{C}} K$ with $K := \ker(\mathrm{Ob}(a,\mathbb{C}) \to G(\mathbb{C}))$.*

Proof. The injective hull $\mathcal{J}$ of $\mathcal{M}$ can be obtained as a limit $\varinjlim \mathcal{J}_\alpha$ with $\mathcal{M} \subseteq \mathcal{J}_\alpha \subseteq \mathcal{J}$ and each $\mathcal{J}_\alpha$ finite artinian. As

$$\mathrm{Ext}^1_S(\Omega^1_{S/\Sigma},\mathcal{M}) \to \varinjlim \mathrm{Ext}^1_S(\Omega^1_{S/\Sigma},\mathcal{J}_\alpha) \cong \mathrm{Ext}^1_S(\Omega^1_{S/\Sigma},\mathcal{J}) = 0$$

is the zero map, there is an index α so that $\mathrm{Ext}^1_S(\Omega^1_{S/\Sigma},\mathcal{M}) \to \mathrm{Ext}^1_S(\Omega^1_{S/\Sigma},\mathcal{J}_\alpha)$ is already zero. Restricting the map ob to $\mathrm{Ext}^1_S(\Omega^1_{S/\Sigma},-)$ yields a commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^1_S(\Omega^1_{S/\Sigma},\mathcal{M}) \arrow[r, "\mathrm{ob}"] \arrow[d, "0"] & \mathrm{Ob}(a,\mathcal{M}) \arrow[r, "\psi"] \arrow[d] & G(\mathcal{M}) \arrow[d, hook] \\
\mathrm{Ext}^1_S(\Omega^1_{S/\Sigma},\mathcal{J}_\alpha) \arrow[r, "\mathrm{ob}"] & \mathrm{Ob}(a,\mathcal{J}_\alpha) \arrow[r, "\psi"] & G(\mathcal{J}_\alpha)\,,
\end{tikzcd}$$

and (1) follows.

In order to derive (2), note that the map ob embeds $T^1_{S/\Sigma}(\mathbb{C})$ into $\mathrm{Ob}(a,\mathbb{C})$, as a is formally versal, and under this map $\mathrm{Ext}^1_S(\Omega^1_{S/\Sigma},\mathbb{C})$ becomes thus isomorphic to a subspace of K by (1). As $\mathrm{Hom}_S(\Omega^1_{S/\Sigma},\mathbb{C})$ is isomorphic to $\mathrm{Ex}(a_0,\mathbb{C})$, the space of infinitesimal deformations, the claim follows from 9.4. $\square$

In practise, it is cumbersome to construct obstruction theories for objects over an arbitrary formal germ S. The following considerations show that it is essentially sufficient to check the existence of obstruction theories over an artinian base.

Definition 6.14. Let $p : \mathbf{E} \to \mathbf{An}_\Sigma$ be a deformation theory. An *obstruction theory for* $\mathbf{E}$ consists in a collection of obstruction theories $\mathrm{Ob}(a,-)$ for every $a \in \mathbf{E}(S)$ over an artinian germ $S \in \mathbf{Art}_\Sigma$ satisfying the following condition.

Ob2: For every inclusion $T \hookrightarrow S$ in $\mathbf{Art}_\Sigma$ and every $a \in \mathbf{E}(S)$ there are functorial isomorphisms

$$\mathrm{Ob}(a \times_S T,\mathcal{M}) \xrightarrow{\cong} \mathrm{Ob}(a,\mathcal{M}), \quad \mathcal{M} \in \mathbf{Coh}(T)\,,$$

such that the diagram

$$\begin{tikzcd} T^1_{T/\Sigma}(\mathcal{M}) \arrow[r, "\mathrm{ob}"] \arrow[d] & \mathrm{Ob}(a\times_S T, \mathcal{M}) \arrow[d, "\cong"] \\ T^1_{S/\Sigma}(\mathcal{M}) \arrow[r, "\mathrm{ob}"] & \mathrm{Ob}(a, \mathcal{M}) \end{tikzcd}$$

commutes.

If $\mathbf{E}$ admits an obstruction theory and if $a \in \mathbf{E}(S)$ over $S \in \hat{\mathbf{An}_\Sigma}$ is a formally versal deformation, then for every module $\mathcal{M} \in \mathbf{Coh}_{art}(S)$ the sequence of groups $\mathrm{Ob}(a_n, \mathcal{M})_{n\gg 0}$, where $a_n := a|S_n$ is the restriction to the n^{th} infinitesimal neighbourhood S_n of $0 \in S$, is essentially constant. Accordingly,

$$\mathrm{Ob}(a, \mathcal{M}) := \mathrm{Ob}(a_n, \mathcal{M}), \quad n \gg 0,$$

constitutes an obstruction theory for a as the sequences

$$\mathrm{Ex}(a_n, \mathcal{M}) \longrightarrow T^1_{S_n/\Sigma}(\mathcal{M}) \longrightarrow \mathrm{Ob}(a_n, \mathcal{M})$$

are exact for $n \gg 0$, and taking the direct limit results in the exact sequence

$$\mathrm{Ex}(a, \mathcal{M}) \cong \varinjlim \mathrm{Ex}(a_n, \mathcal{M}) \longrightarrow T^1_{S/\Sigma}(\mathcal{M}) \cong \varinjlim T^1_{S_n/\Sigma}(\mathcal{M}) \longrightarrow \mathrm{Ob}(a, \mathcal{M})\,.$$

Assume now that $\mathbf{E}$ admits an obstruction theory and that for every $a \in \mathbf{E}(S)$ over an artinian germ S there is a transformation $\mathrm{Ob}(a, -) \to G(a, -)$ into a left exact functor $G(a, -)$ on $\mathbf{Coh}_{art}(S)$. Furthermore, suppose that for every inclusion $T \hookrightarrow S$ in $\mathbf{Art}_\Sigma$ there is an isomorphism $G(a \times_S T, \mathcal{M}) \xrightarrow{\cong} G(a, \mathcal{M})$, $\mathcal{M} \in \mathbf{Coh}\, T$ such that the diagram

$$\begin{tikzcd} \mathrm{Ob}(a\times_S T, \mathcal{M}) \arrow[r] \arrow[d, "\cong"] & G(a\times_S T, \mathcal{M}) \arrow[d, "\cong"] \\ \mathrm{Ob}(a, \mathcal{M}) \arrow[r] & G(a, \mathcal{M}) \end{tikzcd}$$

commutes. Under these assumptions we get the following corollary.

Corollary 6.15. *If $a \in \hat{\mathbf{E}}$ is a formally versal deformation of a_0, then*

$$\dim S \geq \dim_{\mathbb{C}} \mathrm{Ex}(a_0, \mathbb{C}) - \dim_{\mathbb{C}} \ker(\mathrm{Ob}(a_0, \mathbb{C}) \to G(a_0, \mathbb{C}))\,.$$

Proof. Indeed, the functor $G(a, \mathcal{M}) := G(a_n, \mathcal{M}), n \gg 0$, is left exact on $\mathbf{Coh}_{art}(S)$ and satisfies the assumptions of 6.13. $\square$

7. Applications

Deformations of modules. Let $f : X \to \Sigma$ be a flat holomorphic map and $0 \in \Sigma$ a fixed point. In the following we will consider deformations of coherent modules on X. These deformations constitute a deformation theory $p : \mathbf{E} \to \mathbf{An}_\Sigma$, where the objects over a germ $S = (S, 0) \in \mathbf{An}_\Sigma$ are coherent S–flat modules $\mathcal{F}$ on $X_S := X \times_\Sigma S$. A morphism into another module $\mathcal{E} \in \mathbf{E}(T)$ is given by a morphism of Σ-germs $g : S \to T$ together with an isomorphism $(\mathrm{id}_X \times g)^*(\mathcal{F}) \xrightarrow{\sim} \mathcal{E}$.

For a germ $S \in \mathbf{An}_\Sigma$ let $f_S : X_S \to S$ denote the projection. There is a canonical map $T^1_{S/\Sigma}(\mathcal{N}) \to T^1_{X_S}(f_S^*\mathcal{N})$ that assigns to an extension S' of S by $\mathcal{N} \in \mathbf{Coh}\, S$ the extension $X_{S'}$ of X_S by $f_S^*\mathcal{N}$. If $\mathcal{F} \in \mathbf{E}(S)$ is a deformation over S then by 4.4 $\langle [X_{S'}], \mathrm{At}(\mathcal{F})\rangle = 0$ if and only if there is a module $\mathcal{F}'$ on $X_{S'}$ that forms an extension of $\mathcal{F}$ by $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{N} := \mathcal{F} \otimes_{f_S^{-1}\mathcal{O}_S} f_S^{-1}\mathcal{N}$; observe that $\mathcal{F} \underline{\otimes}_{\mathcal{O}_{X_S}} f_S^*\mathcal{N} \cong \mathcal{F} \otimes_{\mathcal{O}_{X_S}} f_S^*\mathcal{N}$

as $\mathcal{F}$ is flat over S. By a standard result in commutative algebra, see [Mat, 7.7], the S'–module $\mathcal{F}'$ is then automatically flat. Hence we obtain the following result.

Lemma 7.1. *The composed maps*

$$\mathrm{ob} : T^1_{S/\Sigma}(\mathcal{N}) \xrightarrow{can} T^1_{X_S}(f_S^*\mathcal{N}) \xrightarrow{\langle *, -\mathrm{At}(\mathcal{F})\rangle} \mathrm{Ext}^2_{X_S}(\mathcal{F}, \mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{N})$$

for $\mathcal{N} \in \mathbf{Coh}\, S$ define an obstruction theory for the deformation theory of coherent modules on X in the sense of 6.10, 6.14.

It is well known that the space of infinitesimal deformations of $\mathcal{F}$ over $S[\mathcal{N}]$ is just $\mathrm{Ext}^1_{X_S}(\mathcal{F}, \mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{N})$. Applying the results of the previous sections we obtain the following theorem.

Theorem 7.2. *Assume that X is proper and smooth over Σ and that $X_0 := f^{-1}(0)$ is bimeromorphically equivalent to a Kähler manifold. Let $\mathcal{F}_0$ be a coherent module on X_0 and let $S = (S, 0) \in \mathbf{An}_\Sigma$ be the basis of a semiuniversal deformation of $\mathcal{F}_0$. If*

$$\sigma : \mathrm{Ext}^2_{X_0}(\mathcal{F}_0, \mathcal{F}_0) \longrightarrow \prod_{n\ge 0} H^{n+2}(X_0, \Omega^n_{X_0})$$

denotes the semiregularity map as in 4.1*, then the following hold.*

1. $\dim S \ge \dim_{\mathbb{C}} \mathrm{Ext}^1_{X_0}(\mathcal{F}_0, \mathcal{F}_0) - \dim_{\mathbb{C}} \ker \sigma$*.*
2. *If σ is injective then S is smooth at* 0 *over a closed subspace of $\hat{\Sigma}$.*

Proof. Let $\mathcal{F}$ be a deformation of $\mathcal{F}_0$ over an artinian germ $T \in \mathbf{Art}_\Sigma$, whence $\mathcal{F}$ is an $\mathcal{O}_{X_T}$–module that is flat over T and restricts to $\mathcal{F}_0$ on X_0. By 4.1, for every coherent module $\mathcal{N}$ on T there is a semiregularity map

$$\sigma_{\mathcal{N}} : \mathrm{Ext}^2_{X_T}(\mathcal{F}, \mathcal{F} \otimes_{\mathcal{O}_T} \mathcal{N}) \longrightarrow \prod_{n\ge 0} H^{n+2}(X_T, \mathcal{N} \otimes_{\mathcal{O}_T} \Omega^n_{X_T/T}),$$

and by 4.7(1), this map is compatible with base change. According to [Del1], the functor

$$\mathcal{N} \longmapsto H^p(X_T, \mathcal{N} \otimes_{\mathcal{O}_T} \Omega^q_{X_T/T}), \quad \mathcal{N} \in \mathbf{Coh}\, T\,,$$

is exact; note that using the results of [Fuj], Deligne's original result extends to the case of compact manifolds that are bimeromorphically equivalent to Kähler manifolds with the same proofs as in (loc.cit.). Applying 6.13, claim (1) follows.

Finally assume that $\sigma = \sigma_{\mathbb{C}}$ is injective. Using induction on the length of $\mathcal{N}$ it follows easily that $\sigma_{\mathcal{N}}$ is injective for all $\mathcal{N} \in \mathbf{Coh}\, T$. In particular, the functor $\mathcal{N} \mapsto \mathrm{Ext}^2_{X_T}(\mathcal{F}, \mathcal{F} \otimes_{\mathcal{O}_T} \mathcal{N})$ is exact on the left and therefore $\mathcal{N} \mapsto \mathrm{Ext}^1_{X_T}(\mathcal{F}, \mathcal{F} \otimes_{\mathcal{O}_T} \mathcal{N})$ is exact on the right. Now 6.7 shows that S is smooth over a closed subspace of $\hat{\Sigma}$, as claimed. □

In case of deformations of modules on a fixed complex space $X = X_0$, that is, when Σ is a reduced point, the result above holds without assuming that X is smooth.

Theorem 7.3. *Let $\mathcal{F}_0$ be a coherent module on X with a finite dimensional space of infinitesimal deformations $\mathrm{Ext}^1_X(\mathcal{F}_0, \mathcal{F}_0)$, and let $S \in \hat{\mathbf{An}}_\Sigma$ be the basis of a formally semiuniversal deformation of $\mathcal{F}_0$. If $\mathcal{F}_0$ has locally finite projective dimension as an $\mathcal{O}_X$–module then the semiregularity map*

$$\sigma : \mathrm{Ext}^2_X(\mathcal{F}_0, \mathcal{F}_0) \longrightarrow \prod_{n\ge 0} H^{n+2}(X, \Lambda^n \mathbb{L}_X)$$

is defined and

$$\dim S \geq \dim_{\mathbb{C}} \operatorname{Ext}^1_X(\mathcal{F}_0, \mathcal{F}_0) - \dim_{\mathbb{C}} \ker \sigma\,.$$

In particular, if σ is injective then S is smooth.

Proof. Let $\mathcal{F}$ be a deformation of $\mathcal{F}_0$ over an artinian germ $T \in \mathbf{An}_\Sigma$, whence $\mathcal{F}$ is an $\mathcal{O}_{X\times T}$–module that is flat over T and induces $\mathcal{F}_0$ on X. The functor

$$\mathcal{N} \longmapsto H^p(X \times T, \mathcal{N} \otimes_{\mathcal{O}_T} \Lambda^q \mathbb{L}_{X\times T/T}) \cong \mathcal{N} \otimes_{\mathbb{C}} H^p(X, \Lambda^q \mathbb{L}_X), \quad \mathcal{N} \in \mathbf{Coh}\, T,$$

is exact. As the semiregularity map is compatible with base change $T \to S$, see 4.7, the result follows as before from 6.13. □

As a special case this contains the result of Artamkin-Mukai, [Art, Muk], that the injectivity of the trace map $\operatorname{Ext}^2_X(\mathcal{F}_0, \mathcal{F}_0) \longrightarrow H^2(X, \mathcal{O}_X)$ implies smoothness of the basis of the semiuniversal deformation of $\mathcal{F}_0$.

The proof of the following variant is similar and left to the reader.

Proposition 7.4. *Let X be a complex space embedded into a complex manifold M. Let $\mathcal{F}_0$ be a coherent $\mathcal{O}_X$–module with $\dim_{\mathbb{C}} \operatorname{Ext}^1_X(\mathcal{F}_0, \mathcal{F}_0) < \infty$. The dimension of the basis $(S, 0)$ of a formally semiuniversal deformation of $\mathcal{F}_0$ satisfies then*

$$\dim S \geq \dim_{\mathbb{C}} \operatorname{Ext}^1_X(\mathcal{F}_0, \mathcal{F}_0) - \ker \dim_{\mathbb{C}} \sigma'\,,$$

where

$$\sigma' : \operatorname{Ext}^2_X(\mathcal{F}_0, \mathcal{F}_0) \xrightarrow{\ can\ } \operatorname{Ext}^2_M(\mathcal{F}_0, \mathcal{F}_0) \xrightarrow{\ \sigma\ } \prod_{n\geq 0} H^{n+2}(M, \Omega^n_M).$$

is the composition of the semiregularity map for $\mathcal{F}_0$ as coherent $\mathcal{O}_M$–module with the canonical map between Ext*–functors. If σ' is injective then S is smooth.* □

Remarks 7.5. 1. If in 7.3 the module $\mathcal{F}$ is supported on a closed subspace Z of X then the map σ factors through a map

$$\sigma_Z : \operatorname{Ext}^2_X(\mathcal{F}_0, \mathcal{F}_0) \longrightarrow \prod_{n\geq 0} H^{n+2}_Z(X, \Lambda^n \mathbb{L}_X)\,,$$

see 4.7 (2). It is clear from the proof that the conclusion of 7.3 also holds with σ_Z instead of σ.

2. Ideally, the map τ in 7.4 should factor through a map

$$\operatorname{Ext}^2_X(\mathcal{F}_0, \mathcal{F}_0) \longrightarrow \prod_{n\geq 0} IH^{n+2,n},$$

where IH denotes intersection cohomology.

The Hilbert scheme. If $f : X \to \Sigma$ is a holomorphic map, let $H_{X/\Sigma}$ be the relative *Douady space* of X that represents the Hilbert functor $\operatorname{Hilb} : \mathbf{An}_\Sigma \to \mathbf{Sets}$, where $\operatorname{Hilb}(S)$ is the set of all closed subspaces of $X_S := X \times_\Sigma S$ that are proper and flat over S. In this section we will give several smoothness criteria for the Douady space that generalize results of [Blo, Kaw1, Ran2].

For this, it is convenient to consider the deformation theory associated to the Hilbert functor. More generally, we will study the deformation theory $p : \mathbf{E} \to \mathbf{An}_\Sigma$, where an object of $\mathbf{E}$ over a germ $S \in \mathbf{An}_\Sigma$ is a subspace $Z \subseteq X_S$ that is flat over S. Note that Z is not required to be proper over S. A morphism of Z into another object, say, $Z' \subseteq X_{S'}$ consists in a morphism $g : S' \to S$ such that $(\mathrm{id}_X \times g)^{-1}(Z) = Z'$. It is well known and easy to see that this constitutes a

deformation theory as explained in Section 6. Let $\mathrm{Ex}(Z/S,\mathcal{N})$, $\mathcal{N} \in \mathbf{Coh}(S)$, be the space of infinitesimal deformations of $Z \subseteq X_S$. The following lemma is well known.

Lemma 7.6. *If $Z \subseteq X_S$ is an S–flat subspace with ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_{X_S}$, then there is an isomorphism $\mathrm{Ex}(Z/S,\mathcal{N}) \cong \mathrm{Hom}_{X_S}(\mathcal{J},\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{N})$, where $\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{N} := \mathcal{O}_Z \otimes_{f_S^{-1}\mathcal{O}_S} f_S^{-1}\mathcal{N}$.* □

Note that $\mathrm{Hom}_{X_S}(\mathcal{J},\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{N})$ is just $T^1_{X/Z}(\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{N})$. In case that Z is locally a complete intersection in X this is just the space of section of the normal bundle $\mathcal{N}_{Z/X}$ of Z in X.

In order to describe an obstruction theory for Z we will assume for simplicity that f is flat; see remark 7.11 (3) for the general case. In the flat case, there is for any coherent $\mathcal{O}_S$–module $\mathcal{N}$ a canonical map $T^1_{S/\Sigma}(\mathcal{N}) \xrightarrow{\alpha} T^1_{X_S/\Sigma}(\mathcal{O}_{X_S} \otimes_{\mathcal{O}_S} \mathcal{N})$ that assigns to an extension $[S']$ of S by $\mathcal{N}$ the extension $[X_{S'}]$ of X by $\mathcal{N}_{X_S} := \mathcal{O}_{X_S} \otimes_{\mathcal{O}_S} \mathcal{N}$. Composing this with the map γ considered in 4.13, we get a map

$$\mathrm{ob} : T^1_{S/\Sigma}(\mathcal{N}) \longrightarrow T^2_{Z/X_S}(\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{N})\,.$$

Applying 4.13, gives the following lemma.

Lemma 7.7. *If $f : X \to \Sigma$ is flat, then the map* ob *constitutes an obstruction theory for Z.* □

Let $f : X \to \Sigma$ be a flat map, $(S, 0)$ a germ over $\Sigma = (\Sigma, 0)$, and let $Z \subseteq X_S$ be an S–flat subspace with special fibre Z_0 over $0 \in S$. If $\mathcal{O}_{Z_0}$ has locally finite projective dimension as a module on $X_0 := f^{-1}(0)$ then by 4.5 the sheaf $\mathcal{O}_Z$ has finite projective dimension over X. Hence applying 4.10 yields a semiregularity map

$$\tau_{\mathcal{N}} : T^2_{Z/X_S}(\mathcal{N}_Z) \longrightarrow \prod_{n\geq 0} H^{n+2}(X_S, \mathcal{N} \underline{\otimes}_{\mathcal{O}_S} \Lambda^n \mathbb{L}_{X_S/S}), \quad \mathcal{N} \in \mathbf{Coh}\, S\,.$$

In case of smooth maps f we get the following result.

Theorem 7.8. *Assume that X is proper and smooth over Σ and that $X_0 := f^{-1}(0)$ is bimeromorphically equivalent to a Kähler manifold. Let $Z_0 \subseteq X_0$ be a closed subspace. If*

$$\tau : T^2_{Z_0/X_0}(\mathcal{O}_{Z_0}) \longrightarrow \prod_{n\geq 0} H^{n+2}(X_0, \Omega^n_{X_0})\ .$$

denotes the semiregularity map for $Z_0 \subseteq X_0$, then the following hold.

1. *If τ is injective then the Douady space $H_{X/\Sigma}$ is smooth over a closed subspace of Σ in a neighbourhood of its point $[Z_0]$.*
2. *The dimension of $H_{X/\Sigma}$ at $[Z_0]$ satisfies*

$$\dim_{[Z_0]} H_{X/\Sigma} \geq \dim_{\mathbb{C}} T^1_{Z_0/X_0}(\mathcal{O}_{Z_0}) - \dim_{\mathbb{C}} \ker \tau\,.$$

The *proof* follows again from 6.13 along the same line of arguments as in 7.2. □

We note that in general the injectivity of τ does not imply that $H_{X/\Sigma}$ is smooth over Σ. For instance, let $X \to \Sigma$ be a smooth family of surfaces over a germ $(\Sigma, 0)$ and consider a curve $C \subseteq X_0$. The semiregularity map $H^1(C,\mathcal{N}_{C/X_0}) \to H^2(X_0, \mathcal{O}_{X_0})$ is dual to the restriction map $H^0(X_0, \omega_{X_0}) \to H^0(C, \omega_{X_0} \otimes_{\mathcal{O}_{X_0}} \mathcal{O}_C)$ as we will show in Section 8. If $X \to \Sigma$ is a versal family of K3-surfaces and C is a connected reduced curve on X_0, then this restriction map is bijective. Hence $H_{X/\Sigma}$

is smooth over a closed subspace of Σ at $[C]$. However, this subspace cannot be all of Σ since there are no curves on the general K3-surface.

In the absolute case, when Σ is a reduced point, the following stronger result holds.

Theorem 7.9. *Let Z be a closed subspace of a complex space X with a finite dimensional space of infinitesimal embedded deformations $T^1_{Z/X}(\mathcal{O}_Z)$ and let $S \in \hat{\mathbf{An}}_\Sigma$ be the basis of a formally semiuniversal deformation of Z. Assume that $\mathcal{O}_Z$ has locally finite projective dimension as $\mathcal{O}_X$–module. With*

$$\tau : T^2_{Z/X}(\mathcal{O}_Z) \longrightarrow \prod_{n\geq 0} H^{n+2}(X, \Lambda^n \mathbb{L}_X)$$

the semiregularity map, the following hold.

1. *The dimension of S satisfies*

$$\dim S \geq \dim_{\mathbb{C}} T^1_{X/Z}(\mathcal{O}_Z) - \dim_{\mathbb{C}} \ker \tau \,.$$

In particular, if τ is injective then S is smooth.

2. *If Z is compact then $\dim_{[Z]} H_X \geq \dim_{\mathbb{C}} T^1_{X/Z}(\mathcal{O}_Z) - \dim_{\mathbb{C}} \ker \tau$. In particular, if τ is injective then H_X is smooth at $[Z]$.* □

In the smooth case this specializes further to the following corollary.

Corollary 7.10. *Let $Z \subseteq X$ be a compact subspace of a complex manifold X. With*

$$\tau : T^2_{Z/X}(\mathcal{O}_Z) \longrightarrow \prod_{n\geq 0} H^{n+2}(X, \Omega^n_X)$$

the semiregularity map, we have $\dim_{[Z]} H_X \geq \dim_{\mathbb{C}} T^1_{Z/X}(\mathcal{O}_Z) - \dim_{\mathbb{C}} \ker \tau$. In particular, if τ is injective then H_X is smooth at $[Z]$. □

As well, the result 7.4 can be formulated in the case of deformations of subspaces. We leave the straightforward formulation and its proof to the reader.

Remarks 7.11. 1. Note that for a locally complete intersection $Z \subseteq X$ with normal bundle $\mathcal{N}_{Z/X}$ there is a canonical isomorphism $T^k_{Z/X}(\mathcal{M}) \cong H^{k-1}(Z, \mathcal{N}_{Z/X} \otimes \mathcal{M})$ for every $\mathcal{O}_Z$–module $\mathcal{M}$ and $k \geq 0$. Hence in this case the statements 7.8–7.10 above hold with $T^2_{Z/X}(\mathcal{O}_Z)$ replaced by $H^1(Z, \mathcal{N}_{Z/X})$.

2. In analogy with 7.5(1), the map τ in 7.10 factors through a map $\tau_Z : T^2_{Z/X}(\mathcal{O}_Z) \to \prod_{n\geq 0} H^{n+2}_Z(X, \Omega^n_X)$, see 4.14 (3). It is clear from the proof that the conclusion of 7.10 remains true for τ_Z instead of τ. A similar remark applies to 7.9.

3. We note that there is also an obstruction theory for embedded deformations if $f : X \to \Sigma$ is not flat. To show this, let S be a space over Σ and $Z \subseteq X_S$ an S–flat subspace. Consider X_S and Z as subspaces of $X \times S$ via the diagonal embedding. For a coherent $\mathcal{O}_S$–module $\mathcal{N}$ there are natural maps

$$T^1_{S/\Sigma}(\mathcal{N}) \xrightarrow{\ \alpha\ } T^1_{X\times S/\Sigma\times\Sigma}(\mathcal{O}_{X\times S} \otimes_{\mathcal{O}_S} \mathcal{N}) \xrightarrow{\ \beta\ } T^1_{X\times S/\Sigma\times\Sigma}(\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{N}) \,,$$

In terms of extensions, if S' is an extension of S by $\mathcal{N}$ then $\alpha([S']) = [X \times S']$ and $\beta([X \times S'])$ is the induced extension of $X \times S$ by $\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{N}$. The embedded deformation $Z \subseteq X_S$ can be extended to an embedded deformation $Z \subseteq X_{S'}$ if and only if $\alpha([X \times S'])$ is in the image of the natural map

$$\gamma : T^1_{Z/\Sigma}(\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{N}) \to T^1_{X\times S/\Sigma\times\Sigma}(\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{N}) \,.$$

This map embeds into a long exact cohomology sequence: namely, if $\mathcal{C}^\bullet$ denotes the mapping cone of $\mathbb{L}_{X\times S/\Sigma\times\Sigma}\underline{\otimes}_{\mathcal{O}_{X\times S}}\mathcal{O}_Z \to \mathbb{L}_{Z/\Sigma}$, then the cokernel of γ embeds into $\mathrm{Ext}^2_Z(\mathcal{C}^\bullet, \mathcal{N}\otimes_{\mathcal{O}_S}\mathcal{O}_Z)$. Thus the composition

$$T^1_{S/\Sigma}(\mathcal{N}) \xrightarrow{\beta\circ\alpha} T^1_{X\times S/\Sigma\times\Sigma}(\mathcal{O}_Z\otimes_{\mathcal{O}_S}\mathcal{N}) \to \mathrm{Ext}^2_Z(\mathcal{C}^\bullet, \mathcal{O}_Z\otimes_{\mathcal{O}_S}\mathcal{N})$$

gives an obstruction theory.

Note that in case of a flat map $f : X \to \Sigma$, the complexes

$$\mathbb{L}_{X_S/S}\underline{\otimes}_{\mathcal{O}_{X_S}}\mathcal{O}_Z \quad \text{and} \quad \mathbb{L}_{X\times S/\Sigma\times\Sigma}\underline{\otimes}_{\mathcal{O}_{X\times S}}\mathcal{O}_Z$$

are quasiisomorphic. Hence $\mathcal{C}^\bullet$ becomes quasiisomorphic to the mapping cone of $\mathbb{L}_{X_S/S}\underline{\otimes}_{\mathcal{O}_{X_S}}\mathcal{O}_Z \to \mathbb{L}_{Z/S}$ and so is quasiisomorphic to $\mathbb{L}_{Z/X_S}$. This is the same obstruction theory we described before.

In the general case, the reader may easily verify that there is a natural map $\mathbb{L}_{Z/X_S} \to \mathcal{C}^\bullet$ that induces a map

$$\mathrm{Ext}^2_Z(\mathcal{C}^\bullet, \mathcal{O}_Z\otimes_{\mathcal{O}_S}\mathcal{N}) \longrightarrow T^2_{Z/X_S}(\mathcal{O}_Z\otimes_{\mathcal{O}_S}\mathcal{N}).$$

Taking the composition with the map τ as defined in 4.10, we arrive at a semiregularity map also in this case.

The Quot–**functor.** Let $f : X \to \Sigma$ be a holomorphic map and $\mathcal{E}$ a coherent sheaf on X. Generalizing partly the results of the previous part we will study the Quot-space $Q_{\mathcal{E}/\Sigma}$ that was constructed as a complex space by Douady in the absolute case and by Pourcin in the relative case, see [Dou, Pou]. We remind the reader that it represents the functor $\mathrm{Quot}_{\mathcal{E}/\Sigma} : \mathbf{An}_\Sigma \to \mathbf{Sets}$, where $\mathrm{Quot}_{\mathcal{E}/\Sigma}(S)$ is the set of all quotients of $\mathcal{E}_S := p_S^*(\mathcal{E})$ that are proper and flat over S; here $p_S : X_S := X\times_\Sigma S \to X$ denotes the projection.

Again it is convenient to consider the associated deformation theory, say, $p : \mathbf{E} \to \mathbf{An}_\Sigma$. An object of $\mathbf{E}$ over a germ $S \in \mathbf{An}_\Sigma$ is a quotient $\mathcal{Q}$ of $\mathcal{E}_S$ that is flat over S. A morphism of $\mathcal{Q}$ into another object, say, $\mathcal{Q}'$ defined over the germ S' is given by a morphism $g : S' \to S$ such that $(\mathrm{id}_X \times g)^*(\mathcal{Q}) = \mathcal{Q}'$ as quotients of $\mathcal{E}_{S'}$. It is well known and easy to see that this constitutes a deformation theory as in Section 3. The space of infinitesimal deformations $\mathrm{Ex}(\mathcal{Q}/S, \mathcal{N})$, $\mathcal{N} \in \mathbf{Coh}(S)$, is described in the following well known lemma.

Lemma 7.12. *If $\mathcal{E}_S \to \mathcal{Q}$ is an S–flat quotient with kernel $\mathcal{F} := \ker(\mathcal{E}_S \to \mathcal{Q})$ then there is a natural isomorphism $\mathrm{Ex}(\mathcal{Q}/S, \mathcal{N}) \cong \mathrm{Hom}_{X_S}(\mathcal{F}, \mathcal{Q}\otimes_{\mathcal{O}_S}\mathcal{N})$.* □

In case $\mathcal{E}$ is flat over Σ, there is furthermore a well known obstruction theory for $\mathcal{Q}$. Since there seems to be no explicit reference for this in the relative case, we describe in brief the construction. First note the following simple lemma whose proof is left to the reader.

Lemma 7.13. *Let X be a complex space and $X \subseteq X'$ an extension of X by a coherent $\mathcal{O}_X$–module $\mathcal{I}$. For coherent $\mathcal{O}_X$–modules $\mathcal{G}$, $\mathcal{H}$ consider the map*

$$\mu : \mathrm{Ext}^1_{X'}(\mathcal{G}, \mathcal{H}) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}\otimes\mathcal{I}, \mathcal{H})$$

that assigns to an X'-extension $[\mathcal{G}'] \in \mathrm{Ext}^1_{X'}(\mathcal{G}, \mathcal{H})$ the homomorphism $\mu(\mathcal{G}')$ obtained from the multiplication map $\mathcal{G}'\otimes\mathcal{I} \to \mathcal{G}'$ through the natural factorization

$$\mathcal{G}'\otimes\mathcal{I} \xrightarrow{proj} \mathcal{G}\otimes\mathcal{I} \xrightarrow{\mu(\mathcal{G}')} \mathcal{H} \hookrightarrow \mathcal{G}'.$$

The map μ is functorial in $\mathcal{G}$ and $\mathcal{H}$ and thus is $\Gamma(X', \mathcal{O}_{X'})$-linear. Moreover, $\ker\mu$ *is canonically isomorphic to* $\mathrm{Ext}^1_X(\mathcal{G}, \mathcal{H})$. □

We note that μ can also be described as the boundary map in the spectral sequence $E_2^{pq} = \mathrm{Ext}^p_X(\mathcal{T}or_q^{\mathcal{O}_{X'}}(\mathcal{G}, \mathcal{O}_X), \mathcal{H}) \Rightarrow \mathrm{Ext}^{p+q}_{X'}(\mathcal{G}, \mathcal{H})$. However, the more explicit description given above is better suited for our needs.

Let us return to the situation as described before 7.13, and consider the composition of the canonical maps

$$T^1_{S/\Sigma}(\mathcal{N}) \xrightarrow{\ \alpha\ } \mathrm{Ext}^1_{X_{S'}}(\mathcal{E}_S, \mathcal{E}_S \otimes_{\mathcal{O}_S} \mathcal{N}) \xrightarrow{\ \beta\ } \mathrm{Ext}^1_{X_{S'}}(\mathcal{E}_S, \mathcal{Q} \otimes_{\mathcal{O}_S} \mathcal{N}),$$

where α maps an extension $[S']$ of S by $\mathcal{N}$ onto the class of $\mathcal{E}_{S'}$. There is furthermore an exact *Ext*–sequence

$$\mathrm{Ext}^1_{X_{S'}}(\mathcal{Q}, \mathcal{Q} \otimes_{\mathcal{O}_S} \mathcal{N}) \xrightarrow{\gamma} \mathrm{Ext}^1_{X_{S'}}(\mathcal{E}_S, \mathcal{Q} \otimes_{\mathcal{O}_S} \mathcal{N}) \xrightarrow{\delta} \mathrm{Ext}^1_{X_{S'}}(\mathcal{F}, \mathcal{Q} \otimes_{\mathcal{O}_S} \mathcal{N})\,.$$

For $\mathcal{G} = \mathcal{E}_S, \mathcal{H} = \mathcal{E}_S \otimes_{\mathcal{O}_S} \mathcal{N}$, the map μ described in 7.13 associates to $\mathcal{E}_{S'}$ just the identity on $\mathcal{E}_S \otimes_{\mathcal{O}_S} \mathcal{N}$. Using again 7.13, the diagram

$$\begin{tikzcd}
\mathrm{Ext}^1_{X_{S'}}(\mathcal{E}_S, \mathcal{E}_S \otimes_{\mathcal{O}_S} \mathcal{N}) \arrow[r, "\delta\circ\beta"] \arrow[d, "\mu"'] & \mathrm{Ext}^1_{X_{S'}}(\mathcal{F}, \mathcal{Q} \otimes_{\mathcal{O}_S} \mathcal{N}) \arrow[d, "\mu"'] \\
\mathrm{Hom}_{X_S}(\mathcal{E}_S \otimes_{\mathcal{O}_S} \mathcal{N}, \mathcal{E}_S \otimes_{\mathcal{O}_S} \mathcal{N}) \arrow[r, "can"] & \mathrm{Hom}_{X_S}(\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{N}, \mathcal{Q} \otimes_{\mathcal{O}_S} \mathcal{N})
\end{tikzcd}$$

commutes, whence the extension $\delta \circ \beta([\mathcal{E}_{S'}])$ maps to 0 under μ and so can be identified with an element of $\mathrm{Ext}^1_{X_S}(\mathcal{F}, \mathcal{Q} \otimes_{\mathcal{O}_S} \mathcal{N})$. In other words, $\delta \circ \beta \circ \alpha$ factors through a map

$$\mathrm{ob} : T^1_{S/\Sigma}(\mathcal{N}) \longrightarrow \mathrm{Ext}^1_{X_S}(\mathcal{F}, \mathcal{Q} \otimes_{\mathcal{O}_S} \mathcal{N})\,.$$

If $\mathrm{ob}([S'])$ vanishes then $\beta([\mathcal{E}_{S'}])$ is in the image of γ, and the corresponding extension $\mathcal{Q}'$ gives a lifting of $\mathcal{Q}$ to S'. This establishes the following result.

Proposition 7.14. *If $\mathcal{E}$ is flat over Σ then the map* ob *just defined provides an obstruction theory for $\mathcal{E}$.* □

It is now immediate how to define a semiregularity map on $\mathrm{Ext}^1_{X_S}(\mathcal{F}, \mathcal{Q} \otimes_{\mathcal{O}_S} \mathcal{N})$.

Definition 7.15. Let $X \to \Sigma$ and $\mathcal{E}$ be as above and let $\mathcal{Q}$ be an S–flat quotient of $\mathcal{E}_S$ over some germ $S = (S, 0)$ over Σ. Assume that $\mathcal{Q}$ has locally finite projective dimension on X. The composition of the boundary map $\mathrm{Ext}^1_{X_S}(\mathcal{F}, \mathcal{Q} \otimes_{\mathcal{O}_S} \mathcal{N}) \to \mathrm{Ext}^2_{X_S}(\mathcal{Q}, \mathcal{Q} \otimes_{\mathcal{O}_S} \mathcal{N})$ with the semiregularity map σ defined in 4.1 yields a map

$$\tau : \mathrm{Ext}^1_{X_S}(\mathcal{F}, \mathcal{Q} \otimes_{\mathcal{O}_S} \mathcal{N}) \longrightarrow \prod_{n \geq 0} H^{n+2}(X_S, \mathcal{N} \otimes_{\mathcal{O}_S} \Lambda^n \mathbb{L}_{X_S/S})$$

that we call the *semiregularity map for $\mathcal{Q}$*.

Now it is possible to deduce results analogous to 7.8–7.10 and 7.4. As a sample we restrict to the following application; the proof is similar to that of 7.8 and left to the reader.

Theorem 7.16. *Assume that X is proper and smooth over Σ and that $X_0 := f^{-1}(0)$ is bimeromorphically equivalent to a Kähler manifold. Let $\mathcal{E}$ be a Σ–flat coherent sheaf on X and $\mathcal{E}_0 \to \mathcal{Q}_0$ be a quotient of $\mathcal{E}_0 := \mathcal{E}|X_0$. With $\mathcal{F}_0$ the kernel of $\mathcal{E}_0 \to \mathcal{Q}_0$ and*

$$\tau : \mathrm{Ext}^1_{X_0}(\mathcal{F}_0, \mathcal{Q}_0) \longrightarrow \prod_{n \geq 0} H^{n+2}(X_0, \Omega^n_{X_0})$$

the semiregularity map for $\mathcal{Q}_0$, the following hold.

1. *If τ is injective then the Quot-space $Q_{\mathcal{E}/\Sigma}$ is smooth over a closed subspace of Σ in a neighbourhood of $[\mathcal{Q}_0]$.*
2. *The dimension of the Quot-space $Q_{\mathcal{E}/\Sigma}$ at the point $[\mathcal{Q}_0]$ satisfies*

$$\dim_{[\mathcal{Q}_0]} Q_{\mathcal{E}/\Sigma} \geq \dim_{\mathbb{C}} \operatorname{Hom}_{X_0}(\mathcal{F}_0, \mathcal{Q}_0) - \dim_{\mathbb{C}} \ker \tau \,.$$

□

Considering a subspace of X as a quotient of $\mathcal{O}_X$, the Douady space becomes a special case of the Quot-space. However, note that 7.16 does not imply the corresponding result for the Douady space. The reason is that the obstruction theory for the Quot-functor does not specialize to the obstruction theory for the Hilb-functor. For instance, if $Z \subseteq X$ is a complete intersection of dimension 0 then $T^2_{Z/X}(\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{N})$ vanishes while $\operatorname{Ext}^1_X(\mathcal{J}, \mathcal{O}_Z)$ is isomorphic to the space of sections of the second exterior power of the normal bundle of Z in X. In general, T^2 provides a much smaller obstruction theory than the Ext^1–functor. Hence it is worthwhile to treat these cases separately.

Remark 7.17. In analogy with 7.11(3), we note that $\mathcal{E}$ need not be flat over Σ in order to obtain an obstruction theory for the Quot–functor. Indeed, with S, $\mathcal{N}$, $\mathcal{Q}$ as in 7.12 and S' an extension of S by $\mathcal{N}$, consider the composition

$$T^1_{S/\Sigma}(\mathcal{N}) \xrightarrow{\alpha} \operatorname{Ext}^1_{X\times S'}(\pi_S^*(\mathcal{E}), \pi_S^*(\mathcal{E}) \otimes_{\mathcal{O}_S} \mathcal{N}) \xrightarrow{\beta} \operatorname{Ext}^1_{X\times S'}(\pi_S^*(\mathcal{E}), \mathcal{Q} \otimes_{\mathcal{O}_S} \mathcal{N})\,,$$

where $\pi_S : X \times S \to S$ denotes the projection. Here α maps an extension $[S']$ of S by $\mathcal{N}$ onto the class of $\pi^*_{S'}(\mathcal{E})$. Now

$$\operatorname{Ext}^1_{X\times S'}(\pi_S^*(\mathcal{E}), \mathcal{Q} \otimes_{\mathcal{O}_S} \mathcal{N}) \cong \operatorname{Ext}^1_{X_{S'}}(\mathcal{E} \,\underline{\otimes}_{\mathcal{O}_\Sigma}\, \mathcal{O}_S, \mathcal{Q} \otimes_{\mathcal{O}_S} \mathcal{N})\,,$$

with $\mathcal{E} \,\underline{\otimes}_{\mathcal{O}_\Sigma}\, \mathcal{O}_S := \pi_S^*(\mathcal{E}) \,\underline{\otimes}_{\mathcal{O}_{X\times S'}}\, \mathcal{O}_{X_{S'}}$. For $\mathcal{Q}$ to be extendable to S', the element $\beta \circ \alpha([S'])$ must be in the image of the map γ in the following exact sequence:

$$\operatorname{Ext}^1_{X_{S'}}(\mathcal{Q}, \mathcal{Q} \otimes_{\mathcal{O}_S} \mathcal{N}) \xrightarrow{\gamma} \operatorname{Ext}^1_{X_{S'}}(\mathcal{E} \,\underline{\otimes}_{\mathcal{O}_\Sigma}\, \mathcal{O}_S, \mathcal{Q} \otimes_{\mathcal{O}_S} \mathcal{N}) \xrightarrow{\delta} \operatorname{Ext}^1_{X_{S'}}(\mathcal{F}^\bullet, \mathcal{Q} \otimes_{\mathcal{O}_S} \mathcal{N})\,,$$

where $\mathcal{F}^\bullet$ is the mapping cone of $\mathcal{E} \,\underline{\otimes}_{\mathcal{O}_\Sigma}\, \mathcal{O}_S[-1] \to \mathcal{Q}[-1]$. As before, one can verify that $\delta \circ \beta \circ \alpha$ factors through a map

$$\mathrm{ob} : T^1_{S/\Sigma}(\mathcal{N}) \longrightarrow \operatorname{Ext}^1_{X_S}(\mathcal{F}^\bullet, \mathcal{Q} \otimes_{\mathcal{O}_S} \mathcal{N})\,,$$

and this map constitutes an obstruction theory. The reader may easily check that this gives rise to a semiregularity map on $\operatorname{Ext}^1_{X_S}(\mathcal{F}^\bullet, \mathcal{Q} \otimes_{\mathcal{O}_S} \mathcal{N})$.

Deformations of mappings. In this part we will generalize the semiregularity map for embedded deformations, see 4.10, to deformations of holomorphic maps $X_0 \to Y_0$. For the special case that X_0 is a stable curve over an algebraic manifold Y_0, such a semiregularity map was independently constructed by K. Behrend and B. Fantechi in order to define refined Gromov-Witten invariants in certain situations. We consider the following setup.

7.18. Let $\pi : Y \to \Sigma = (\Sigma, 0)$ be a fixed germ of a flat holomorphic mapping, set $Y_0 := \pi^{-1}(0)$ and let $f_0 : X_0 \to Y_0$ be a morphism of complex spaces with X_0 compact. By a deformation of X_0/Y_0 over a germ $(S, 0) \in \mathbf{An}_\Sigma$ we mean a

commutative diagram

$$
\text{(1)} \qquad \begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "q"'] & & Y_S := Y \times_\Sigma S \arrow[dl, "\pi_S"] \\ & S & \end{tikzcd}
$$

such that q is flat and proper and f induces f_0 on the special fibre $X_0 = q^{-1}(0)$. This deformation is, abusively, denoted by X/Y_S. Such deformations form in a natural way a deformation theory over $\mathbf{An}_\Sigma$.

It is well known, and follows easily from the existence of the (relative) Douady space, that there are always convergent versal deformations of X_0/Y_0, see [Fle2, BKo].

The infinitesimal deformations of X_0/Y_0 can be described as follows. Let X/Y_S be a deformation of X_0/Y_0 over S as in (1) and let $\mathcal{N}$ be a coherent $\mathcal{O}_S$–module. An $S' := S[\mathcal{N}]$-extension of $f : X \to Y_S$ by $\mathcal{N}$ consists in a deformation $X'/Y_{S'}$ of X_0/Y_0 over S' as in the diagram

$$
\text{(2)} \qquad \begin{tikzcd} X' \arrow[rr, "f'"] \arrow[dr] & & Y_{S'} \arrow[dl, "\pi_{S'}"] \\ & S' = S[\mathcal{N}] & \end{tikzcd}
$$

that induces on S the given deformation X/Y_S as in (1). Let $\operatorname{Ex}(X/Y_S, \mathcal{N})$ denote the group of these extensions. An $S[\mathcal{N}]$-extension $X'/Y_{S[\mathcal{N}]}$ of X/Y_S corresponds to an extension of X by $\mathcal{N}_X := \mathcal{N} \otimes_{\mathcal{O}_S} \mathcal{O}_X$ that is a space over Y_S. Hence we obtain the following lemma.

Lemma 7.19. *There is a canonical isomorphism* $\operatorname{Ex}(X/Y_S, \mathcal{N}) \cong T^1_{X/Y_S}(\mathcal{N}_X)$. $\square$

Next we will describe obstructions for extending deformations. Let X/Y_S be as in (1) and let $S \hookrightarrow S'$ be an extension of S by $\mathcal{N}$. The extension $Y_{S'}$ of Y_S by $\mathcal{N}_{Y_S} := \mathcal{N} \otimes_{\mathcal{O}_S} \mathcal{O}_{Y_S}$ gives an element $[Y_{S'}]$ of $T^1_{Y_S}(\mathcal{N}_{Y_S})$. The composition of the two canonical maps

$$
\gamma : T^1_{Y_S}(\mathcal{N}_{Y_S}) \longrightarrow \operatorname{Ext}^1_X(Lf^*(\mathbb{L}_{Y_S/S}), \mathcal{N}_X) \longrightarrow T^2_{X/Y_S}(\mathcal{N}_X)
$$

admits the following interpretation in terms of extensions.

Lemma 7.20. *The class of the extension $Y_{S'}$ is mapped to* 0 *under γ if and only if there is an extension X' of X by $\mathcal{N}_X$ together with a map $f' : X' \to Y_{S'}$ that induces f on Y and the identity on $\mathcal{N}_X$. In particular, the composite map*

$$
\mathrm{ob} : T^1_{S/\Sigma}(\mathcal{N}) \xrightarrow{can} T^1_{Y_S}(\mathcal{N}_{Y_S}) \xrightarrow{\gamma} T^2_{X/Y_S}(\mathcal{N}_X)
$$

provides an obstruction theory in the sense of 6.10*.*

Proof. To prove this statement, we use the existence of tangent functors $T^i_f(-)$ for holomorphic mappings as constructed in [Fle1], see also [Ill]. These functors fit into an exact sequence

$$
(*) \qquad \cdots \longrightarrow T^1_f(\mathcal{N}_{Y_S}) \xrightarrow{\beta} T^1_{Y_S}(\mathcal{N}_{Y_S}) \xrightarrow{\gamma} T^2_{X/Y_S}(\mathcal{N}_X) \longrightarrow \cdots ,
$$

see [Fle1, 3.4]. The group $T^1_f(\mathcal{N}_{Y_S})$ is canonically isomorphic to the set of all isomorphism classes of extensions of f by $\mathcal{N}_{Y_S}$. Such extension is a holomorphic

map $f' : X' \to Y'$, where X', Y' are extensions of X, Y_S by $\mathcal{N}_X$, $\mathcal{N}_{Y_S}$, respectively, with f' inducing the map f on X and the identity on $\mathcal{N}_X$; see [Fle1, 3.16] for details. Moreover, the map β in $(*)$ assigns to $[f']$ the extension $[Y']$. In view of the exactness of the sequence $(*)$ this proves the lemma. □

In a next step, we generalize 4.9(1) to arbitrary mappings.

Proposition 7.21. *For every morphism of complex spaces $f : X \to Y$ and every complex of $\mathcal{O}_X$–module $\mathcal{M}$ bounded below there are canonical maps*

$$T^k_{X/Y}(\mathcal{M}) \longrightarrow \mathrm{Ext}^k_Y(Rf_*(\mathcal{O}_X), Rf_*(\mathcal{M})), \quad k \in \mathbb{Z}\,.$$

In case $\mathcal{M} = \mathcal{O}_X$, this specializes to a map of graded Lie algebras $T^\bullet_{X/Y}(\mathcal{O}_X) \to \mathrm{Ext}^\bullet_Y(Rf_*(\mathcal{O}_X), Rf_*(\mathcal{O}_X))$.

Proof. Let $(X_*, W_*, \mathcal{R}_*)$ be a resolvent for X over Y and let $\mathcal{M}_* \to \tilde{\mathcal{M}}_*$ be a quasiisomorphism into a W_*-acyclic complex $\tilde{\mathcal{M}}_*$ of $\mathcal{O}_{X_*}$–modules as in 2.15. We need to construct a map

$$T^k_{X/Y}(\mathcal{M}) \cong H^k(\mathrm{Hom}_{\mathcal{R}_*}(\Omega^1_{\mathcal{R}_*/Y}, \tilde{\mathcal{M}}_*)) \longrightarrow \mathrm{Ext}^k_Y(Rf_*(\mathcal{O}_X), Rf_*(\mathcal{M}))\,.$$

Let $f^{-1}(\mathcal{O}_Y)$ be the topological preimage of the sheaf $\mathcal{O}_Y$ and let $f_*^{-1}(\mathcal{O}_Y)$ be the associated simplicial sheaf of rings on X_*. As the topological restriction $\mathcal{R}_*|X_*$ is a sheaf of abelian groups on X_*, we can form its associated Čech complex and so we can consider the composed map

$$(*) \qquad \begin{aligned} &\mathrm{Hom}_{\mathcal{R}_*}(\Omega^1_{\mathcal{R}_*/Y}, \tilde{\mathcal{M}}_*) \hookrightarrow \mathrm{Der}_Y(\mathcal{R}_*, \tilde{\mathcal{M}}_*) \\ &\quad \hookrightarrow \mathrm{Hom}_{f_*^{-1}(\mathcal{O}_Y)}(\mathcal{R}_*, \tilde{\mathcal{M}}_*) \longrightarrow \mathrm{Hom}_{f^{-1}(\mathcal{O}_Y)}(C^\bullet(\mathcal{R}_*|X_*), C^\bullet(\tilde{\mathcal{M}}_*)), \end{aligned}$$

where the first two maps are the natural inclusions and the last one is given by the Čech functor. There is always a natural morphism $\mathrm{Hom}(-,-) \to \mathrm{RHom}(-,-)$, thus taking cohomology we obtain a natural map

$$(**) \qquad T^k_{X/Y}(\mathcal{M}) \longrightarrow \mathrm{Ext}^k_{f^{-1}(\mathcal{O}_Y)}(C^\bullet(\mathcal{R}_*|X_*), C^\bullet(\tilde{\mathcal{M}}_*)) \cong \mathrm{Ext}^k_{f^{-1}(\mathcal{O}_Y)}(\mathcal{O}_X, \mathcal{M})\,,$$

where the final isomorphism results from the fact that the complexes $C^\bullet(\mathcal{R}_*|X_*)$ and $C^\bullet(\tilde{\mathcal{M}}_*)$ are quasiisomorphic to $\mathcal{O}_X$, resp. $\mathcal{M}$, see 2.28. Composing $(**)$ with

$$Rf_* : \mathrm{Ext}^k_{f^{-1}(\mathcal{O}_Y)}(\mathcal{O}_X, \mathcal{M}) \longrightarrow \mathrm{Ext}^k_{\mathcal{O}_Y}(Rf_*(\mathcal{O}_X), Rf_*(\mathcal{M}))$$

gives the desired map. If $\mathcal{M} = \mathcal{O}_X$, then replacing $\tilde{\mathcal{M}}$ by $\mathcal{R}_*$ and $C^\bullet(\tilde{\mathcal{M}}_*)$ by $C^\bullet(\mathcal{R}_*|X_*)$, the first inclusion in $(*)$ becomes bijective, and so the argument shows the compatibility with the Lie algebra structure. □

In the next lemma we provide a criterion as to when $Rf_*(\mathcal{O}_X)$ is a perfect complex on Y_S. We keep the notation introduced in 7.18.

Lemma 7.22. *If Y is smooth over Σ, then $Rf_*(\mathcal{O}_X)$ is a perfect complex on Y_S.*

Proof. For every coherent $\mathcal{O}_S$–module $\mathcal{N}$ the natural map

$$Rf_*(\mathcal{O}_X) \underline{\otimes} \mathcal{N} \to Rf_*(\mathcal{N}_X)$$

is a quasiisomorphism and so the claim follows from 4.5. □

We are now able to apply the constructions of the previous section to define a semiregularity map for deformations of mappings. Let $Y \to \Sigma$ and $f : X \to Y_S$ be as in 7.18 and assume moreover that $Y \to \Sigma$ is smooth. We define a semiregularity map

$$\tau_{\mathcal{N}} : T^2_{X/Y_S}(\mathcal{N}_X) \to \prod_{p \geq 0} H^{p+2}(Y_S, \mathcal{N} \otimes \Omega^p_{Y_S/S})$$

as the composition of the two maps

$$T^2_{X/Y_S}(\mathcal{N}_X) \xrightarrow{can} \operatorname{Ext}^2_{Y_S}(Rf_*(\mathcal{O}_X), Rf_*(\mathcal{O}_X) \underline{\otimes} \mathcal{N}) \xrightarrow{\sigma} \prod_{p \geq 0} H^{p+2}(Y_S, \mathcal{N} \otimes \Omega^p_{Y_S/S}),$$

where the first map is as in 7.21 and σ is the semiregularity map defined in 4.1.

In analogy with 7.9 we are now able to deduce the following result.

Theorem 7.23. *Assume that $\pi : Y \to \Sigma$ is proper and smooth and that $Y_0 := f^{-1}(0)$ is bimeromorphically equivalent to a Kähler manifold. Let $f_0 : X_0 \to Y_0 := \pi^{-1}(0)$ be a proper holomorphic map and denote $S = (S, 0)$ the basis of the semi-universal deformation of X_0/Y. If*

$$\tau_0 : T^2_{X_0/Y_0}(\mathcal{O}_{X_0}) \longrightarrow \prod_{p \geq 0} H^{p+2}(Y_0, \Omega^p_{Y_0})$$

is the semiregularity map as above, then the following hold.

1. $\dim S \geq \dim_{\mathbb{C}} T^1_{X_0/Y_0}(\mathcal{O}_{X_0}) - \dim_{\mathbb{C}} \ker \tau_0$.
2. *If τ_0 is injective, then S is smooth at 0 over a closed subspace of Σ.*

Remarks 7.24. 1. In the special case that Σ is a reduced point, $Y = Y_0$ is a compact complex manifold of dimension d, and $f : X \to Y$ is a map from a rational curve X into Y, the group $T^2(X/Y, \mathcal{O}_X)$ is isomorphic to the group $H^1(X, f^*(\Theta_Y))$, where Θ_Y is the tangent bundle of Y. Thus the top component of the semiregularity map provides a map

$$H^1(X, f^*(\Theta_Y)) \longrightarrow H^d(Y, \Omega^{d-2}_Y),$$

and we recover thus the map constructed by Behrend and Fantechi.

2. The results 7.3 and 7.4 also admit generalizations to the case of deformations of mappings. We leave the straightforward formulation and proof to the reader.

8. Comparison with Bloch's semiregularity map

8.1. Assume that X is a compact complex manifold and $Z \subseteq X$ is a locally complete intersection of (constant) codimension q with ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_X$. In this section we will compare our semiregularity map from 4.10 with the semiregularity map defined in [Blo]. Observe that for a locally complete intersection $T^k_{Z/X}(\mathcal{O}_Z) \cong H^{k-1}(Z, \mathcal{N}_{Z/X})$ for all k, where $\mathcal{N}_{Z/X} \cong (\mathcal{J}/\mathcal{J}^2)^\vee := \mathcal{H}\mathrm{om}_{\mathcal{O}_Z}(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_Z)$ is the normal bundle of Z in X. Bloch's semiregularity map is constructed as follows. With $m := \dim Z$, there is a natural pairing

$$\Omega^{m+1}_X \times \Lambda^{q-1}\mathcal{N}^\vee_{Z/X} \xrightarrow{1 \times \Lambda^{q-1}\bar{d}} \Omega^{m+1}_X \times \Omega^{q-1}_X \otimes \mathcal{O}_Z \xrightarrow{\wedge} \omega_X \otimes \mathcal{O}_Z,$$

where $\bar{d} : \mathcal{N}^\vee_{Z/X} \cong \mathcal{J}/\mathcal{J}^2 \to \Omega^1_X \otimes \mathcal{O}_Z$ is the map induced by the differential $d : \mathcal{J} \to \Omega^1_X$. Equivalently, this amounts to a map

$$\Omega^{m+1}_X \longrightarrow \Lambda^{q-1}\mathcal{N}_{Z/X} \otimes \omega_X \cong \mathcal{N}^\vee_{Z/X} \otimes \omega_Z, \tag{1}$$

where we have used the adjunction formula $\omega_Z \cong \det \mathcal{N}_{Z/X} \otimes \omega_X$ and the isomorphism $\Lambda^{q-1}\mathcal{N}_{Z/X} \cong \mathcal{N}^\vee_{Z/X} \otimes \det \mathcal{N}_{Z/X}$. Dualizing the induced map in cohomology $H^{m-k}(X, \Omega^{m+1}_X) \to H^{m-k}(Z, \mathcal{N}^\vee_{Z/X} \otimes \omega_Z)$ gives a map

$$\tau_B : H^k(Z, \mathcal{N}_{Z/X}) \longrightarrow H^{q+k}(X, \Omega^{q-1}_X), \tag{2}$$

and Bloch's semiregularity map is just this map for $k = 1$. We will compare it with the component

$$\tau : H^k(Z, \mathcal{N}_{Z/X}) \cong T^{k+1}_{Z/X}(\mathcal{O}_Z) \longrightarrow H^{q+k}(X, \Omega^{q-1}_X)$$

of our semiregularity map defined in 4.10.

Proposition 8.2. *The maps τ_B and τ coincide.*

For the proof, we need a more explicit description of the semiregularity map. First observe that

$$H^p\left(X, \mathcal{H}^q_Z(\Omega^{q-1}_X)\right) \xrightarrow{\cong} H^{q+p}_Z(X, \Omega^{q-1}_X)\ , \quad p \geq 0\,,$$

as $\mathcal{H}^i_Z(\Omega^{q-1}_X) = 0$ for $i \neq q$; for the algebraic case, see [Blo], whereas in the analytic case this follows from [Sche]. To proceed further we describe the local cohomology sheaves $\mathcal{H}^q_Z(\Omega^{q-1}_X)$ in terms of a Cousin-type complex.

8.3. Let $\mathcal{E}$ be a vector bundle on X and let Z be as above. Assume $U \subseteq X$ is an open subset such that the ideal $\mathcal{J} \subseteq \mathcal{O}_U$ of $Z \cap U \subseteq U$ is generated by sections $f_1, \ldots, f_q \in \Gamma(U, \mathcal{O}_U)$. For an index $\alpha = (\alpha_1, \ldots, \alpha_p)$ with $1 \leq \alpha_1 < \ldots < \alpha_p \leq q$ set $|\alpha| = p$ and $U_\alpha := \{x \in U | f_\alpha(x) \neq 0\}$, where $f_\alpha = \prod_{i=1}^p f_{\alpha_i}$. Consider basis elements $\delta f_\alpha := \delta f_{\alpha_1} \wedge \cdots \wedge \delta f_{\alpha_p}$ and the Cousin complex

$$\mathcal{C}^\bullet_Z(\mathcal{E}|U): \qquad 0 \to \mathcal{E}|U =: (\mathcal{E}|U_\emptyset)\delta f_\emptyset \to \prod_{|\alpha|=1} \mathcal{E}_\alpha \delta f_\alpha \to \cdots \to \prod_{|\alpha|=q} \mathcal{E}_\alpha \delta f_\alpha \to 0\,,$$

where $\mathcal{E}_\alpha := j_{\alpha *}(\mathcal{E}|U_\alpha)$ with j_α the inclusion $U_\alpha \hookrightarrow U$. The differential on $\mathcal{C}^\bullet_Z(\mathcal{E}|U)$ is given by $\partial(\delta f_\alpha) := -\sum_{i=1}^q \delta f_i \wedge \delta f_\alpha$. The minus sign occurs in order to have an exact sequence of complexes

$$0 \to \mathcal{C}^\bullet(\{U_i \cap U\}, \mathcal{E})[-1] \to \mathcal{C}^\bullet_Z(\mathcal{E}|U) \to \mathcal{E}|U \to 0\,,$$

where $\mathcal{C}^\bullet := \mathcal{C}^\bullet(\{U_i \cap U\}, \mathcal{E})$ is the sheafified Čech complex; observe that the differential on $\mathcal{C}^\bullet[-1]$ is the *negative* of the differential on $\mathcal{C}^\bullet$! It is well known that $\mathcal{H}^p_Z(\mathcal{E}|U) \cong \mathcal{H}^p(\mathcal{C}^\bullet_Z(\mathcal{E}|U))$. Note that $\mathcal{C}^\bullet_Z(\mathcal{O}_U)$ carries a graded algebra structure via $\delta f_\alpha \delta f_\beta := \delta f_\alpha \wedge \delta f_\beta$ and that $\mathcal{C}^\bullet_Z(\mathcal{E}|U)$ is a graded module over $\mathcal{C}^\bullet_Z(\mathcal{O}_U)$.

In [Blo, p.61], Bloch defines a natural map

$$\mu : \mathcal{N}_{Z/X} \longrightarrow \mathcal{H}^q_Z(X, \Omega^{q-1}_X)$$

that is locally given as follows. Multiplying the element

$$\omega := \left[\frac{\delta f_1}{f_1} \wedge \cdots \wedge \frac{\delta f_q}{f_q}\right] \in \Gamma\left(U, \mathcal{H}^q_Z(\mathcal{O}_X)\right)$$

with $df_1 \wedge \cdots \wedge df_q$ gives a form $\omega \otimes df_1 \wedge \cdots \wedge df_q$ in $H^0\left(U, \mathcal{H}^q_Z(X, \Omega^q_X)\right)$ that is independent of the choice of the equations $f_1, \ldots, f_q$. The map μ is then given by contraction against $\omega \otimes df_1 \wedge \cdots \wedge df_q$, which yields for a linear map $\varphi : \mathcal{J} \longrightarrow \mathcal{O}_Z$ the explicit expression

$$\mu(\varphi) = \sum (-1)^{i-1} \omega \cdot \varphi(f_i) \otimes df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_q\,. \tag{3}$$

Now τ_B in 8.1 (2) is the composition

$$H^k(Z,\mathcal{N}_{Z/X}) \xrightarrow{\mu} H^k\left(X, \mathcal{H}^q_Z(X,\Omega^{q-1}_X)\right) \cong H^{q+k}_Z(X,\Omega^{q-1}_X) \xrightarrow{can} H^{q+k}(X,\Omega^{q-1}_X)\,,$$

see (loc.cit.) for a proof. To compare this map with our semiregularity map, observe first that τ also admits a factorization

$$H^k(Z,\mathcal{N}_{Z/X}) \cong T^{k+1}_{Z/X}(\mathcal{O}_Z) \longrightarrow H^{q+k}_Z(X,\Omega^{q-1}_X) \xrightarrow{can} H^{q+k}(X,\Omega^{q-1}_X)\ ,$$

see 4.7. As taking traces is compatible with localization, to deduce 8.2 it is sufficient to show the following lemma.

Lemma 8.4. *The diagram of $\mathcal{O}_Z$-modules*

$$\begin{tikzcd}
\mathcal{E}xt^1_X(\mathcal{O}_Z,\mathcal{O}_Z) \cong \mathcal{N}_{Z/X} \arrow[rr, "*\cdot(-\mathrm{At}(\mathcal{O}_Z))^{q-1}/(q-1)!"] \arrow[dr, "\mu"'] & & \mathcal{E}xt^q_X(\mathcal{O}_Z,\mathcal{O}_Z\otimes\Omega^{q-1}_X) \arrow[dl, "\mathrm{Tr}"] \\
& \mathcal{H}^q_Z(X,\Omega^{q-1}_X) &
\end{tikzcd}$$

commutes.

Proof. This is a local calculation, so we may suppose X Stein and Z defined by equations $f_1,\dots,f_q$ so that the Koszul complex $K_\bullet(\underline{f},\mathcal{O}_X)$ is an $\mathcal{O}_X$-resolution of the sheaf $\mathcal{O}_Z$. More explicitly, set

$$K^{-p} := K_p(\underline{f},\mathcal{O}_X) = \bigoplus_{i_1<\dots<i_p} \mathcal{O}_X\gamma f_{i_1}\wedge\dots\wedge\gamma f_{i_p}\ ,$$

with the Koszul differential given by $\partial(\gamma f_j) = f_j$. Note that $K^\bullet$ is the free graded algebra over $\mathcal{O}_X$ with generators $\gamma f_i \in K^{-1}$, $1 \le i \le q$, in (cohomological) degree -1 and that ∂ is a derivation of degree 1. In particular,

$$\mathrm{Ext}^k_X(\mathcal{O}_Z,\mathcal{O}_Z \underline{\otimes}_{\mathcal{O}_X} \mathcal{M}) \cong H^k(\mathrm{Hom}_X(K^\bullet, K^\bullet\otimes_{\mathcal{O}_X}\mathcal{M})), \quad k\ge 0\,,$$

for every coherent $\mathcal{O}_X$-module $\mathcal{M}$. Consider on $K^\bullet$ the connection

$$\nabla : K^\bullet \longrightarrow K^\bullet\otimes\Omega^1_X \quad \text{with} \quad \nabla\left(\gamma f_{i_1}\wedge\dots\wedge\gamma f_{i_p}\right) = 0\ .$$

The Atiyah class of $\mathcal{O}_Z$ is now the element of $\mathrm{Ext}^1_X(\mathcal{O}_Z,\mathcal{O}_Z\otimes\Omega^1_X)$ represented by

$$[\partial,\nabla] : K^\bullet \longrightarrow K^\bullet\otimes\Omega^1_X\,,$$

an $\mathcal{O}_X$-linear map of degree 1. Note that ∇ as well as $[\partial,\nabla]$ are derivations on the ring $K^\bullet$ and that $[\partial,\nabla](\gamma f_i) = -1\otimes df_i$. Thus, the map

$$[\partial,\nabla]^{q-1}/(q-1)! : K^\bullet \longrightarrow K^\bullet\otimes\Omega^{q-1}_X\,,$$

which is only nonzero on K^{-q+1} and K^{-q}, is given there by

$$\begin{aligned}
\gamma f_1\wedge\dots\wedge\widehat{\gamma f_i}\wedge\dots\wedge\gamma f_q &\longmapsto (-1)^{\binom{q}{2}} 1\otimes(df_1\wedge\dots\wedge\widehat{df_i}\wedge\dots\wedge df_q)\\
\gamma f_1\wedge\dots\wedge\gamma f_q &\longmapsto \sum_i (-1)^{\binom{q-1}{2}+i}\gamma f_i\otimes df_1\wedge\dots\wedge\widehat{df_i}\wedge\dots\wedge df_q\,.
\end{aligned}$$

Consider now $\bar\varphi \in \mathrm{Hom}_X(\mathcal{J},\mathcal{O}_Z) \cong \mathrm{Ext}^1_X(\mathcal{O}_Z,\mathcal{O}_Z)$ and write $\bar\varphi(f_i) = \bar\varphi_i$ with sections φ_i of $\mathcal{O}_X$. Under the isomorphism $\mathrm{Ext}^1_X(\mathcal{O}_Z,\mathcal{O}_Z) \cong H^1(\mathrm{Hom}_X(K^\bullet,K^\bullet))$, the element $\bar\varphi$ corresponds to the derivation $\varphi : K^\bullet \to K^\bullet$ with $\varphi(\gamma f_i) = \varphi_i$. The

composition $\varphi \circ [\partial, \nabla]^{q-1}/(q-1)!$ is a map of degree q and is therefore determined by the component

$$\begin{aligned} \varphi \circ [\partial, \nabla]^{q-1}/(q-1)! : K^{-q} &\longrightarrow K^0 \otimes \Omega_X^{q-1} \\ \gamma f_1 \wedge \cdots \wedge \gamma f_q &\longmapsto \sum (-1)^{\binom{q-1}{2}+i} \varphi_i \otimes df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_q \ . \end{aligned}$$

Note that $\varphi \circ [\partial, \nabla]^{q-1}$ represents $\langle [\bar{\varphi}], \mathrm{At}^{q-1}(\mathcal{O}_Z) \rangle$.

Now we wish to take traces to produce an element in $\mathcal{H}_Z^q(X, \Omega_X^{q-1})$. Let $\mathcal{C}_Z^\bullet = \mathcal{C}_Z^\bullet(\mathcal{O}_X)$ be the Cousin complex, see 8.3. Let $\hat{\gamma} f_i$ be the basis of $\mathrm{Hom}_X(K^{-1}, \mathcal{O}_X)$ dual to γf_i and set

$$\hat{\gamma} f_\alpha := \hat{\gamma} f_{\alpha_1} \wedge \cdots \wedge \hat{\gamma} f_{\alpha_p} = (-1)^{\binom{p}{2}} \hat{\gamma} f_{\alpha_p} \wedge \cdots \wedge \hat{\gamma} f_{\alpha_1}.$$

With this convention, $(-1)^{\binom{p}{2}} \hat{\gamma} f_\alpha$ is the basis element dual to $\gamma f_\alpha := \gamma f_{\alpha_1} \wedge \cdots \wedge \gamma f_{\alpha_p}$ and the differential on $\mathcal{H}om_X(K^\bullet, \mathcal{O}_X)$ is given by multiplying from the left with $-\sum f_i \cdot \hat{\gamma} f_i$. The computation above gives that

$$\varphi \circ [\partial, \nabla]^{q-1}/(q-1)! = \hat{\gamma} f_1 \wedge \cdots \wedge \hat{\gamma} f_q \sum (-1)^{q+i} \varphi_i \otimes df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_q \ .$$

Using 8.5 below, this is mapped under the trace map to

$$(-1)^{q-1} \frac{\delta f_1 \wedge \cdots \wedge \delta f_q}{f_1 \cdots f_q} \cdot \sum (-1)^{i-1} \varphi_i \otimes df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_q \ .$$

Comparing with (3) above, the result follows. □

It remains to verify the following lemma.

Lemma 8.5. *The trace map* $\mathcal{E}xt_X^q(\mathcal{O}_Z, \mathcal{O}_Z) \longrightarrow \mathcal{H}_Z^q(\mathcal{O}_X)$ *maps the class of* $\hat{\gamma} f_1 \wedge \cdots \wedge \hat{\gamma} f_q \in \mathrm{Hom}_X(K^{-q}, \mathcal{O}_X) \subseteq \mathrm{Hom}_X(K^\bullet, K^\bullet)$ *onto the class of*

$$\frac{\delta f_1 \wedge \cdots \wedge \delta f_q}{f_1 \cdots f_q} \ .$$

Proof. As $\mathcal{C}_Z^\bullet$ is a sheaf of flat $\mathcal{O}_X$–modules, the complex $K^\bullet \otimes \mathcal{C}_Z^\bullet$ is quasiisomorphic to $\mathcal{O}_Z \otimes \mathcal{C}_Z^\bullet \cong \mathcal{O}_Z$. Therefore, the canonical projection $\mathcal{C}_Z^\bullet \to \mathcal{C}_Z^0 = \mathcal{O}_X$ induces a quasiisomorphism

$$K^\bullet \otimes \mathcal{C}_Z^\bullet \longrightarrow K^\bullet \ .$$

In a first step, we construct a section of this projection,

$$\begin{tikzcd} K^\bullet \arrow[rr, "\psi"] \arrow[dr, "\mathrm{id}"'] & & K^\bullet \otimes \mathcal{C}_Z^\bullet \arrow[dl, "proj"] \\ & K^\bullet & \end{tikzcd}$$

as follows. The map $\hat{\gamma} f_\alpha \mapsto \delta f_\alpha / f_\alpha$, where $\hat{\gamma} f_\alpha = \hat{\gamma} f_{\alpha_1} \wedge \cdots \wedge \hat{\gamma} f_{\alpha_p}$ is as above, realizes $\mathcal{H}om_X(K^\bullet, \mathcal{O}_X)$ as a subcomplex of $\mathcal{C}_Z^\bullet$. We define ψ to be the composition

$$K^\bullet \longrightarrow \mathcal{H}om_X(K^\bullet, K^\bullet) \cong K^\bullet \otimes \mathcal{H}om_X(K^\bullet, \mathcal{O}_X) \hookrightarrow K^\bullet \otimes \mathcal{C}_Z^\bullet \ ,$$

where the first map is given by $k \mapsto k \cdot \mathrm{id}$. As the identity corresponds to the element $\sum (-1)^{\binom{|\alpha|}{2}} \gamma f_\alpha \otimes \hat{\gamma} f_\alpha$ in $K^\bullet \otimes \mathcal{H}om_X(K^\bullet, \mathcal{O}_X)$, the map ψ is given explicitly by

$$k \longmapsto k \cdot \sum_\alpha (-1)^{\binom{|\alpha|}{2}} \gamma f_\alpha \otimes \delta f_\alpha / f_\alpha \ .$$

Now we can define the local trace map $\mathcal{H}om_X(K^\bullet, K^\bullet) \to \mathcal{C}_Z^\bullet$ as the composition of

$$\mathcal{H}om_X(K^\bullet, K^\bullet) \cong \mathcal{H}om_X(K^\bullet, \mathcal{O}_X) \otimes K^\bullet \xrightarrow{1\otimes\psi} \mathcal{H}om_X(K^\bullet, \mathcal{O}_X) \otimes K^\bullet \otimes \mathcal{C}_Z^\bullet \xrightarrow{\operatorname{Tr}\otimes 1} \mathcal{C}_Z^\bullet.$$

The image of $\hat\gamma f_1 \wedge \cdots \wedge \hat\gamma f_q$ under these maps is just given by

$$\begin{aligned} \hat\gamma f_1 \wedge \cdots \wedge \hat\gamma f_q &\overset{1\otimes\psi}{\longmapsto} \hat\gamma f_1 \wedge \cdots \wedge \hat\gamma f_q \otimes \sum_\alpha (-1)^{\binom{|\alpha|}{2}} \gamma f_\alpha \otimes \delta f_\alpha / f_\alpha \\ &\overset{\operatorname{Tr}\otimes 1}{\longmapsto} \frac{\delta f_1 \wedge \cdots \wedge \delta f_q}{f_1 \cdots f_q}, \end{aligned}$$

as desired. □

8.6. Another application of the above construction concerns the infinitesimal Abel-Jacobi map. Let X be an n–dimensional compact algebraic manifold and $Z \subseteq X$ a closed submanifold of codimension q. The infinitesimal Abel-Jacobi map is the differential at $[Z]$ of the Abel-Jacobi map

$$\operatorname{Hilb}_X \longrightarrow J^q(X)$$

into the intermediate Jacobian $J^q(X)$; see, for example, [Gre, Gri2], or [Voi] in the analytic case. This differential can be considered as a map

$$\beta : H^0(Z, \mathcal{N}_{Z/X}) \longrightarrow H^q(X, \Omega_X^{q-1})$$

that has the same homological description in terms of Serre duality as Bloch's semiregularity map and is just the map τ_B in 8.1 (2) for $k = 0$, see (loc.cit.).

Applying 8.2, we obtain the following description of the infinitesimal Abel-Jacobi map.

Proposition 8.7. *The infinitesimal Abel-Jacobi map fits into the commutative diagram*

$$\begin{tikzcd} H^0(Z, \mathcal{N}_{Z/X}) \arrow[r, "\cong"] \arrow[d, "\beta"'] & \operatorname{Ext}^1_X(\mathcal{O}_Z, \mathcal{O}_Z) \arrow[d, "\langle *, (-\operatorname{At}(\mathcal{O}_Z))^{q-1}/(q-1)!\rangle"] \\ H^q(X, \Omega_X^{q-1}) & \operatorname{Ext}^q_X(\mathcal{O}_Z, \mathcal{O}_Z \otimes \Omega_X^{q-1}). \arrow[l, "\operatorname{Tr}"'] \end{tikzcd}$$

□

This statement should generalize. There should be an Abel-Jacobi map for deformations of arbitrary coherent sheaves on a compact algebraic manifold such that its differential is essentially given by multiplication with powers of the Atiyah class as above. More precisely, let us pose the following problem.

Problem 8.8. For a coherent sheaf $\mathcal{F}_0$ on a compact algebraic manifold X the Chern character $\operatorname{ch}_k(\mathcal{F}_0)$ is a well defined class in the Chow group $CH^k(X)_\mathbb{Q}$. Assume that the sheaf $\mathcal{F}$ on $X \times S$ is a semiuniversal deformation of $\mathcal{F}_0$ over a germ $S = (S, 0)$. If $\mathcal{F}_s$ denotes the restriction of $\mathcal{F}$ to the fibre $X \cong X \times \{s\}$, then the following should hold:

1. The map $s \longmapsto \operatorname{ch}_k(\mathcal{F}_s)$ provides a family of k-dimensional cycles on X.
2. Integrating over a (topological) $(2k+1)$-chain in X with boundary $\operatorname{ch}_k(\mathcal{F}_s) - \operatorname{ch}_k(\mathcal{F}_0)$ gives a holomorphic map $S \xrightarrow{\varphi_k} J^k(X)$.

3. The derivative of φ_k is given by

$$T_0 S \xrightarrow{\cong} \operatorname{Ext}^1_X(\mathcal{F}_0, \mathcal{F}_0) \xrightarrow{\operatorname{Tr}\langle *, (-\operatorname{At}(\mathcal{F}_0))^k/k!\rangle} H^{k+1}(X, \Omega^k_X)\,.$$

9. Appendix: Infinitesimal deformations and integral dependence

We recall first the definition of integral dependence, see [ZSa1]. Let R be a ring and $I \subseteq R$ an ideal. An element $x \in R$ is *integral* over I if there is an equation

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

with $a_\nu \in I^\nu$.

For instance, every element of I is integral over I. The set $\overline{I} \subseteq R$ of all elements from R that are integral over I is an ideal, the *integral closure* of I in R.

We remind the reader of the following criterion for integral dependence that we formulate for our purposes as follows. Let k be an algebraically closed field and let $A = (A, \mathfrak{m})$ be a local noetherian complete k-algebra with residue field k. If $I \subseteq \mathfrak{m}$ is an ideal, then f is in the integral closure of I if and only if for every "arc" $\alpha : A \to k[\![T]\!]$ the element $\alpha(f)$ is contained in $\alpha(I)k[\![T]\!]$.

9.1. For the main result of this section we consider the following setup. Let $\Lambda \to A$ be a morphism of complete noetherian k-algebras with residue field k and assume that $A \cong R/I$ with $R := \Lambda[\![X_1, \dots, X_s]\!]$ and $I \subseteq \mathfrak{m}_\Lambda R + (X_1, \dots, X_s)^2$. It is well known that for every finite A–module M there is a canonical isomorphism

$$(*) \qquad T^1_{A/\Lambda}(M) \cong \operatorname{coker}\Big(\operatorname{Hom}_A(\Omega^1_{A/\Lambda}, M) \longrightarrow \operatorname{Hom}_A(I, M)\Big)\,,$$

where $T^1_{A/\Lambda}(M)$ is the first tangent cohomology. The elements of $T^1_{A/\Lambda}(M)$ correspond to isomorphism classes of algebra extensions $[A']$ of A by M. On the level of such algebra extensions the isomorphism above is given as follows. If $[A'] \in T^1_{A/\Lambda}$ is an algebra extension of A by M, let $p' : R \to A'$ be a morphism of Λ–algebras lifting the given map $p : R \to A$. Restricting p' to I gives a map $\varphi_{A'} := p'|I : I \to M$, and the correspondence $(*)$ assigns to $[A']$ the residue class of this homomorphism. As $I \subseteq \mathfrak{m}_\Lambda R + (X_1, \dots, X_s)^2$, we get in particular that

$$(**) \qquad T^1_{A/\Lambda}(k) \cong \operatorname{Hom}_A(I, k)\,.$$

There is always a canonical inclusion of $\operatorname{Ext}^1_A(\Omega^1_{A/\Lambda}, M)$ into $T^1_{A/\Lambda}(M)$. In case $M = k$, another important subspace of $T^1_{A/\Lambda}(k)$ is $\operatorname{Ex}^c_{A/\Lambda}(k)$, the space of *curvilinear extensions*. This is by definition the subspace generated by all curvilinear extensions $[A']$, which are those extensions that fit into a commutative diagram of Λ–algebras

$$(\mathbf{D}) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & k & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0 \\ & & \cong\big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & k & \longrightarrow & k[\![t]\!]/(t^{n+1}) & \longrightarrow & k[\![t]\!]/(t^n) & \longrightarrow & 0\,. \end{array}$$

We will give the following characterizations of these subspaces.

Theorem 9.2. 1. *If k is algebraically closed then under the isomorphism $(**)$ the subspace $\operatorname{Ex}^c_{A/\Lambda}(k)$ of $T^1_{A/\Lambda}(k)$ corresponds to the subspace $\operatorname{Hom}_A(I/J, k)$ of $\operatorname{Hom}_A(I, k)$, where J is the integral closure of $\mathfrak{m}I$ in I.*

2. *The elements of* $\mathrm{Ext}^1_A(\Omega^1_{A/\Lambda}, M) \hookrightarrow T^1_{A/\Lambda}(M)$ *correspond to those extensions* $[A']$ *of* $[A]$ *by* M *for which the associated Jacobi map* $j : M \to \Omega^1_{A'/\Lambda} \otimes_{A'} A$ *is injective.*

3. *If* $\mathrm{Char}\, k = 0$ *and* $\Lambda = M = k$*, then there are inclusions*

$$\mathrm{Ex}^c_{A/k}(k) \subseteq \mathrm{Ext}^1_A(\Omega^1_{A/k}, k) \subseteq T^1_{A/k}(k)\,.$$

Proof. For (1), let A' be a curvilinear extension of A by k and let $p' : R \to A'$ be a morphism of Λ–algebras lifting the given map $p : R \to A$, so that $\varphi_{A'} = p'|I : I \to k$ corresponds under $(**)$ to the extension $[A']$. By the valuative criterion of integral dependence mentioned above, J is in the kernel of p', whence $\varphi_{A'} \in \mathrm{Hom}_A(I/J, k)$. Thus $\mathrm{Ex}^c_{A/\Lambda}(k) \subseteq \mathrm{Hom}_A(I/J, k)$. To show equality, assume that α is a k-linear form on $\mathrm{Hom}_A(I, k)$ that vanishes on $\mathrm{Ex}^c_{A/\Lambda}(k)$. We need to show that α vanishes on $\mathrm{Hom}_A(I/J, k)$. Such a linear form can be written as $\mathrm{Hom}_A(I, k) \ni f \mapsto \alpha(f) = f(x)$ for some $x \in I$. By assumption, for every curvilinear extension A' of A by k, the element $\alpha(\varphi_{A'}) = \varphi_{A'}(x)$ vanishes. Applying the valuative criterion of integral dependence it follows that $x \in J$ and so α vanishes on $\mathrm{Hom}_A(I/J, k)$, as desired.

In order to deduce (2) note that the map $\mathrm{Ext}^1_A(\Omega^1_{A/\Lambda}, M) \to T^1_{A/\Lambda}(M)$ assigns to an extension $0 \to M \to E \xrightarrow{q} \Omega^1_{A/\Lambda} \to 0$ the algebra extension $[A']$ of A by M that is the quotient of the trivial extension $A[E]$ by the ideal $\ker((d, -q) : A[E] \to \Omega^1_{A/\Lambda})$, where d is the differential. The reader may easily verify that then $E \cong \Omega^1_{A'/\Lambda} \otimes_{A'} A$ and that the Jacobi map j becomes the inclusion of M into E, whence j is injective. Conversely, if for an extension $[A']$ the map j is injective, then it is easily seen that the construction just described recovers $[A']$ from the extension $E := \Omega^1_{A'/\Lambda} \otimes_{A'} A$ of $\Omega^1_{A/\Lambda}$ by M.

Finally for (3), if $[A']$ is a curvilinear extension as in the diagram (**D**) in 9.1, then the composed map $k \xrightarrow{j} \Omega^1_{A'/\Lambda} \otimes_{A'} A \to k[\![t]\!]/(t^n) \cdot dt$ is the map $1 \mapsto d(t^n)$ and so is injective. Hence j is also injective, proving the inclusion $\mathrm{Ex}^c_{A/k}(k) \subseteq \mathrm{Ext}^1_A(\Omega^1_{A/k}, k)$. □

The following result shows how to bound the dimension of A in terms of its curvilinear extensions, as

$$\mathrm{Ex}^c_{A/\Lambda}(k) \cong \mathrm{Hom}_A(I/J, k) \cong \mathrm{Hom}_k(I/(J + \mathfrak{m}I), k)$$

by the preceding result. Kawamata [Kaw2] attributes the corresponding geometric argument to Mori.

Proposition 9.3. *Let* $A = R/I$ *be a quotient of a local ring* $(R, \mathfrak{m}, k)$ *modulo an ideal* $I \subseteq \mathfrak{m}$*. If* $J \subseteq I$ *is integral over* $\mathfrak{m}I$*, then*

$$\dim A \geq \dim R - \dim_k(I/(J + \mathfrak{m}I))\,.$$

Proof. Replacing J by $J + \mathfrak{m}I$ we may assume that $J \supseteq \mathfrak{m}I$. Choose elements $x_1, \ldots, x_s \in I$ that form a basis of the k-vector space I/J and consider the natural ring homomorphism

$$k[X_1, \ldots, X_s] \longrightarrow \bigoplus_{\nu=0}^{\infty} (I^\nu/\mathfrak{m}I^\nu)\, T^\nu = R[IT]/\mathfrak{m}R[IT]$$

given by $X_i \mapsto \bar{x}_i T \in (I/\mathfrak{m}I)T$. In a first step we prove that this map is finite. In fact, the elements $\bar{f}T\,, f \in J$, generate the ring $R[IT]/\mathfrak{m}R[IT]$ as an algebra

over $k[X_1, \dots, X_s]$, and if $f^n + a_1 f^{n-1} + \cdots + a_n = 0$ is an equation of integral dependence for such an $f \in J$ over $\mathfrak{m}I$, the coefficients satisfy $a_\nu \in (\mathfrak{m}I)^\nu$, whence $(\bar{f}T)^n = 0$ and finiteness follows. This implies

$$\dim R[IT]/\mathfrak{m}R[IT] \leq s\,. \tag{1}$$

As $R[IT]/\mathfrak{m}R[IT]$ appears as the special fibre of

$$R/I \longrightarrow G_I(R) := \bigoplus_{\nu=0}^{\infty} I^\nu/I^{\nu+1}\,,$$

we obtain

$$\dim R[IT]/\mathfrak{m}R[IT] \geq \dim G_I(R) - \dim R/I = \dim R - \dim R/I\,,$$

see [Mat, Thms.15.1, 15.7]. Together with (1) the result follows. □

For the next result we have to assume that the ground field k has characteristic zero.

Proposition 9.4. *If $\Lambda \to A$ is a morphism of local noetherian complete k–algebras with residue fields k, then*

$$\dim A \geq \dim_k \operatorname{Hom}_A(\Omega^1_{A/\Lambda}, k) - \dim_k \operatorname{Ext}^1_A(\Omega^1_{A/\Lambda}, k)\,.$$

Proof. In the absolute case, where $\Lambda = k$, this is a result due to Scheja and Storch, see [SSt, 3.5]. Alternatively, it follows from the chain of inequalities

$$\begin{aligned}\dim A &\geq \dim R - \dim_k(I/J + \mathfrak{m}I) = \dim_k \operatorname{Hom}_A(\Omega^1_{A/k}, k) - \operatorname{Ex}^c_{A/k}(k)\\ &\geq \dim_k \operatorname{Hom}_A(\Omega^1_{A/k}, k) - \operatorname{Ext}^1_A(\Omega^1_{A/k}, k),\end{aligned}$$

where we have applied 9.3 and 9.2.

To deduce the general case, set $\bar{A} := A/\mathfrak{m}_\Lambda A$. The spectral sequence

$$E_2^{pq} = \operatorname{Ext}^p_{\bar{A}}(\operatorname{Tor}^A_q(\Omega^1_{A/\Lambda}, \bar{A}), k) \Rightarrow \operatorname{Ext}^{p+q}_A(\Omega^1_{A/\Lambda}, k)$$

yields

$$\operatorname{Hom}_A(\Omega^1_{A/\Lambda}, k) \cong \operatorname{Hom}_{\bar{A}}(\Omega^1_{\bar{A}/k}, k) \quad \text{and} \quad \operatorname{Ext}^1_A(\Omega^1_{A/\Lambda}, k) \supseteq \operatorname{Ext}^1_{\bar{A}}(\Omega^1_{\bar{A}/k}, k)\,.$$

Hence the result follows from the chain of inequalities

$$\begin{aligned}\dim A \geq \dim \bar{A} &\geq \dim_k \operatorname{Hom}_{\bar{A}}(\Omega^1_{\bar{A}/k}, k) - \dim_k \operatorname{Ext}^1_{\bar{A}}(\Omega^1_{\bar{A}/k}, k)\\ &\geq \dim_k \operatorname{Hom}_A(\Omega^1_{A/\Lambda}, k) - \dim_k \operatorname{Ext}^1_A(\Omega^1_{A/\Lambda}, k).\end{aligned}$$

□

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